1

MATRICES AND DETERMINANTS AND ECONOMIC APPLICATIONS

1.1 Introduction

The study of matrices and determinants is of immense significance in business and economics. We find the applications of matrices and determinants in various topics of economics and management such as linear programming, theory of games, general equilibrium analysis, matrix multiplier, input–output analysis and so. With the help of matrices and determinants, we can solve equations and system of equations. In chapter, we shall discuss matrices and determinants.

1.2 Martrices : Definition and Notations

Definition of Matrix : A matrix is a rectangular array of numbers. In otherwords, a set of mn numbers arranged in the form of rectangular array of m rows and n column is called m × n matrix (to be read as 'm' by 'n' matrix). The general form of a matrix of order m × n is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & & a_{mn} \end{bmatrix}_{m \times n}$$

The numbers a_{11}, a_{12}, a_{22}.....a_{ij} etc. are called the elements of the matrix A. m is the number of rows and n is the number of columns.

For example : $A = \begin{pmatrix} 2 & 1 & 3 & 4 \\ 3 & 5 & 5 & 5 \\ 2 & 4 & 6 & 6 \end{pmatrix}_{3 \times 4}$ is a matrix of order 3 × 4

1.3 Notations of Matrices : Matrices are denoted by capital letters A, B, C or X, Y, Z etc. and its elements by small letters a, b, c..... The elements of the matrix are enclosed by any of the brackets i.e. [ ], ( ), { }. The position of the elements of a matrix is indicated by the subscripts attached to the element. For example, a_{12} indicates that the element lies in the first row and second column i.e. first subscript denotes row and second denotes column.

1.4 Order of a Matrix : The number of rows and columns of a matrix determines the order of the matrix. Thus, a matrix, having m rows and n column is said to be of the order m × n (read as 'm' by 'n'). In particular, a matrix having 3 rows and 4 columns is of the order 3 × 4 and it is called a 3 × 4 matrix.

For example : $A = \begin{pmatrix} 3 & 5 & 6 \\ 2 & 4 & 8 \end{pmatrix}_{2 \times 3}$

is a matrix of order 2 × 3 since there are two rows and three columns.

Types of Matrices

(1) Rectangular Matrix : A matrix in which the number of rows and columns are not equal is called a rectangular matrix. For example,

$$A = \begin{bmatrix} 2 & 5 & 1 \\ 3 & 4 & 1 \end{bmatrix}_{2\times 3}$$ is a rectangular matrix of order 2 x 3.

(2) Square Matrix : A matrix in which the number of rows is equal to the number of columns is called a square matrix. For example,

$$A = \begin{pmatrix} 2 & 1 & 5 \\ 3 & 2 & 4 \\ 3 & 3 & 6 \end{pmatrix}$$

is a square matrix of order 3×3 i.e. of order 3.

Note : The elements 2, 2, 6 in this matrix are called **diagonal elements** and the line along which they lie is called the **principal diagonal.**

(3) Diagonal Matrix : A square matrix in which all diagonal elements are not zero and all non diagonal elements are zeros is called a diagonal matrix. For example,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}_{3\times 3}$$

is a diagonal matrix of order 3×3

(4) Scalar Matrix : A diagonal matrix in which diagonal elements are equal (but not equal to 1), is called a scalar matrix. For example,

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

is a scalar matrix of order 3×3.

(5) Identity (or Unit) Matrix : A square matrix whose each diagonal element is unity and all other elements are zeros is called an identity (or unit) matrix. An identity matrix of order 3 is denoted by I_3 or simply by I. For example,

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a unit matrix of order 3.

(6) Null (zero matrix or) Matrix : A matrix of any order (rectangular or square) whose each of its element is zero is called a null matrix (or a zero matrix) and is denoted by O. For example,

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } O = \begin{pmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \end{pmatrix}$$

are null matrices of order 2×2 and 2×3 respectively.

(7) Row Matrix : A matrix having only one row and any number of columns is called a row matrix (or a row vector). For example,

$A = [1, 2, 3]$ is a row matrix of order 1×3

(8) Column Matrix : A matrix having only one column and any number of rows is called a column matrix (or a column vector). For example,

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ is a column matrix of order } 3 \times 1$$

(9) Upper Triangular and Lower Triangular Matrix : A square matrix is called an upper triangular matrix if all the elements below the principal diagonal are zero and it is said to be lower triangular matrix if all the elements above the principal diagonal are zero. For example,

$$A = \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 4 \\ 0 & 0 & 8 \end{pmatrix} \text{ is an upper triangular matrix}$$

and $$B = \begin{pmatrix} 7 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 3 \end{pmatrix} \text{ is a lower triangular matrix}$$

(10) Sub Matrix : A matrix obtained by deleting some rows or column or both of a given matrix is called its sub matrix. For example,

$$\begin{pmatrix} 4 & 6 \\ 5 & 2 \end{pmatrix}_{2\times2} \text{ is a sub matrix of } \begin{pmatrix} 4 & 5 & 6 \\ 3 & 3 & 6 \\ 5 & 2 & 1 \end{pmatrix}_{3\times3}$$

The sub matrix of the order 2×2 is obtained by deleting 2nd row and 3rd column of matrix A.

(11) Trace of a Matrix : In a square matrix A, the sum of all the diagonal elements is called the trace of A. It is denoted by tr A. For example,

$$\text{If } A = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 5 & 2 & 3 \end{pmatrix}_{3\times3}, \text{ then tr } A = 2 + 1 + 3 = 6$$

$$\text{In general, tr } (A) = a_{11} + a_{22} + a_{33} + \ldots\ldots a_{nn} = \sum_{i=1}^{n} a_{ii}$$

(12) Equality of Two Matrices : Two matrices A and B are said to be equal if and only if they are of the same order and each element of A is equal to the corresponding element of B. For example,

$$\text{If } A = \begin{pmatrix} 3 & 4 \\ 0 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 4 \\ 0 & -1 \end{pmatrix}, \text{ then } A = B$$

since the matrix A is of the order 2 x 2 and the matrix B is of the order 2x2 and also the corresponding elements of A and B are equal.

Operations on Matrices (or Matrix Operations)

Addition (or Subtraction) of Matrices

The addition (or subtraction) of two or more matrices is possible only if these matrices have the same dimension i.e. the matrices must have the same number of rows and same number of columns. The sum (or difference) of matrices is obtained by adding (or subtracting) the corresponding elements of the given matrices. For example

$$\text{If } A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 4 & 8 \\ 3 & -2 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 5 \\ 0 & 1 & 6 \end{bmatrix}$$

then,

$$A + B = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 4 & 8 \\ 3 & -2 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 5 \\ 0 & 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4 & 3+1 & 4+0 \\ -2+1 & 4+3 & 8+5 \\ 3+0 & -2+1 & -1+6 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ -1 & 7 & 13 \\ 3 & -1 & 5 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 4 & 8 \\ 3 & -2 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 5 \\ 0 & 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1-4 & 3-1 & 4-0 \\ -2-1 & 4-3 & 8-5 \\ 3-0 & -2-1 & -1-6 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 4 \\ -3 & 1 & 3 \\ 3 & -3 & -7 \end{bmatrix}$$

Multiplication of a Matrix by a Scalar

The multiplication of a matrix by a scalar K implies multiplication of every element of the matrix A by K.

If A = [*aij*] is any matrix of dimension m × n and K is any scalar, then the multiplication of the matrix A by the scalar K is denoted by KA and is obtained by multiplying each element of A by the scalar K i.e.

$KA = K\,[aij]_{m\times n} = [K\,aij]_{m\times n}$

For example, if, $A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix}$ and K = 3, find 3A.

$$3A = 3\begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 2\times 3 & 3\times 3 & 1\times 3 \\ 0\times 3 & -1\times 3 & 5\times 3 \end{pmatrix} = \begin{pmatrix} 6 & 9 & 3 \\ 0 & -3 & 15 \end{pmatrix}$$

Example 1. If $A = \begin{pmatrix} 3 & 8 & 11 \\ 6 & -3 & 8 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -6 & 15 \\ 3 & 8 & 17 \end{pmatrix}$

Compute 7A + 5B

Solution : We have $A = \begin{pmatrix} 3 & 8 & 11 \\ 6 & -3 & 8 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -6 & 15 \\ 3 & 8 & 17 \end{pmatrix}$

$$\therefore \quad 7A = \begin{pmatrix} 7\times 3 & 7\times 8 & 7\times 11 \\ 7\times 6 & 7\times -3 & 7\times 8 \end{pmatrix} = \begin{pmatrix} 21 & 56 & 77 \\ 42 & -21 & 56 \end{pmatrix}$$

$$5B = \begin{pmatrix} 5\times 1 & 5\times -6 & 5\times 15 \\ 5\times 3 & 5\times 8 & 5\times 17 \end{pmatrix} = \begin{pmatrix} 5 & -30 & 75 \\ 15 & 40 & 85 \end{pmatrix}$$

$$\therefore 7A + 5B = \begin{pmatrix} 21 & 56 & 77 \\ 42 & -21 & 56 \end{pmatrix} + \begin{pmatrix} 5 & -30 & 75 \\ 15 & 40 & 85 \end{pmatrix}$$

$$= \begin{pmatrix} 21+5 & 56-30 & 77+75 \\ 42+15 & -21+40 & 56+85 \end{pmatrix} = \begin{pmatrix} 26 & 26 & 152 \\ 57 & 19 & 141 \end{pmatrix}$$

Example 2. If $A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{pmatrix}$

Find $3A - 4B$

Solution : $3A - 4B = 3\begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 6 & 9 & 3 \\ 0 & -3 & 15 \end{pmatrix} - \begin{pmatrix} 4 & 8 & -24 \\ 0 & -4 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 6-4 & 9-8 & 3-(-24) \\ 0-0 & -3-(-4) & 15-12 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 27 \\ 0 & 1 & 3 \end{pmatrix}$$

Example 3. Find a 2×4 matrix X, such that
$2A + 3X = 5B$ where

$$A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 4 & 8 & -1 & 5 \end{pmatrix}, B = \begin{pmatrix} -2 & 7 & 5 & 0 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

Solution : We have $2A + 3X = 5B$
or $3X = 5B - 2A$

$$\text{or} \quad X = \frac{1}{3}[5B - 2A]$$

$$\text{Given :} \quad A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 4 & 8 & -1 & 5 \end{pmatrix}, B = \begin{pmatrix} -2 & 7 & 5 & 0 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

$$5B - 2A = 5\begin{pmatrix} -2 & 7 & 5 & 0 \\ 1 & 3 & 2 & 4 \end{pmatrix} - 2\begin{pmatrix} 1 & 3 & 0 & 2 \\ 4 & 8 & -1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} -12 & 29 & 25 & -4 \\ -3 & -1 & 12 & 10 \end{pmatrix}$$

$$\text{Thus,} \quad X = \frac{1}{3} \cdot [5B - 2A] = \frac{1}{3}\begin{pmatrix} -12 & 29 & 25 & -4 \\ -3 & -1 & 12 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 29/3 & 25/3 & -4/3 \\ -1 & -1/3 & 4 & 10/3 \end{pmatrix}$$

Example 4. If $A = \begin{bmatrix} 4 & 2 & 3 & 1 \\ 5 & 1 & 2 & 0 \\ 3 & 4 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -1 & 4 & 5 \\ -1 & 2 & 1 & 7 \\ -1 & -3 & 0 & 9 \end{bmatrix}$

Compute $A + 2B + C$

Solution : $2B = 2\begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 0 & 3\\0 & 0 & 1 & 6\end{bmatrix} = \begin{bmatrix}2 & 4 & 6 & 8\\0 & 2 & 0 & 6\\0 & 0 & 2 & 12\end{bmatrix}$

$$A + 2B + C = \begin{bmatrix}4 & 2 & 3 & 1\\5 & 1 & 2 & 0\\3 & 4 & 1 & 3\end{bmatrix} + \begin{bmatrix}2 & 4 & 6 & 8\\0 & 2 & 0 & 6\\0 & 0 & 2 & 12\end{bmatrix} + \begin{bmatrix}2 & -1 & 4 & 5\\-1 & 2 & 1 & 7\\-1 & -3 & 0 & 9\end{bmatrix}$$

$$= \begin{bmatrix}8 & 5 & 13 & 14\\4 & 5 & 3 & 13\\2 & 1 & 3 & 24\end{bmatrix}$$

Example 5. Solve the equation :

$$X + \begin{bmatrix}1 & 5 & -2\\7 & 2 & -9\\3 & -1 & 5\end{bmatrix} = \begin{bmatrix}2 & -3 & 4\\7 & -1 & 2\\8 & -2 & 2\end{bmatrix}$$

Solution : The given equation is :

$$X + \begin{bmatrix}1 & 5 & -2\\7 & 2 & -9\\3 & -1 & 5\end{bmatrix} = \begin{bmatrix}2 & -3 & 4\\7 & -1 & 2\\8 & -2 & 2\end{bmatrix}$$

$$\therefore \quad X = \begin{bmatrix}2 & -3 & 4\\7 & -1 & 2\\8 & -2 & 2\end{bmatrix} - \begin{bmatrix}1 & 5 & -2\\7 & 2 & -9\\3 & -1 & 5\end{bmatrix}$$

$$= \begin{bmatrix}2-1 & -3-5 & 4+2\\7-7 & -1-2 & 2+9\\8-3 & -2+1 & 2-5\end{bmatrix} = \begin{bmatrix}1 & -8 & 6\\0 & -3 & 11\\5 & -1 & -3\end{bmatrix}$$

Example 6. If $A = \begin{pmatrix}5 & -3 & -6\\-2 & 3 & 7\end{pmatrix}$

Find matrix X such that X + A = 0 where O is the null matrix.

Solution : Given X + A = 0

$\Rightarrow$ X = O – A, where 0 is a null matrix

$$\therefore X = \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\end{pmatrix} - \begin{pmatrix}5 & -3 & -6\\-2 & 3 & 7\end{pmatrix}$$

$$= \begin{pmatrix}-5 & 3 & 6\\2 & -3 & -7\end{pmatrix}$$

Properties of Matrix Addition

If A, B and C are any three matrices of same dimension, then

(1) Matrix addition is commutative i.e.

$A + B = B + A$

(2) Matrix addition is associative i.e.

$(A + B) + C = A + (B + C)$

(3) Matrix addition is distributive w.r.t. scalar K. i.e.

$K(A + B) = KA + KB$

$$= \begin{pmatrix} 21 & -7 \\ -7 & 28 \end{pmatrix} \qquad \text{(ii)}$$

From (i) and (ii) $7(A + B) = 7A + 7B$

EXERCISE 1.1

1. (a) Define the terms : (i) Scalar matrix (ii) Diagonal matrix (iii) Square matrix (iv) Equality of two matrices (v) Null matrix (vi) Identity matrix (vii) Matrix (viii) Trace of matrix (ix) Sub–matrix of a matrix (x) upper triangular and lower triangular matrix.
   (b) When is matrix addition defined ?
2. (a) State whether the following matrices are equal or not

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

   (b) Find a, b, c, d where $\begin{bmatrix} a+b & c+d \\ c-d & a-b \end{bmatrix} = \begin{bmatrix} 15 & -2 \\ 6 & 3 \end{bmatrix}$

3. For each of the following matrices, find A + B, A – B, 3A – 2B

$$A = \begin{pmatrix} 2 & -3 & 5 \\ 6 & 7 & -3 \\ 2 & -1 & 0 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 & 5 \\ 6 & 7 & 8 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Evaluate :

$$\left[\begin{pmatrix} 2 & 3 & 4 \\ -1 & 0 & 2 \\ 5 & 6 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 2 \\ -1 & 2 & -3 \\ 6 & 7 & -8 \end{pmatrix}\right] + \begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & 7 \\ -6 & 3 & 2 \end{pmatrix}$$

5. If $A = \begin{pmatrix} 2 & -5 & 1 \\ -2 & -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 4 & 0 \\ 5 & -2 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 7 & -6 & 2 \\ 1 & -4 & 11 \end{pmatrix}$

Find : (i) A + B (ii) A – B (iii) 2 A + B – C (iv) 3A – 4B (v) 4B – 4C

6. (a) Find the trace of the matrix :

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

(b) Find a 2 × 4 matrix X such that :

A – 2X = 3B if

$A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 0 & 3 \\ 1 & -2 & 2 & 3 \end{pmatrix}$

(c) If $A = \begin{pmatrix} 2 & -2 & 7 \\ 4 & 6 & 3 \end{pmatrix}$, find matrix X such that X + A = 0 where 0 is a null matrix.

7. If $A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 7 \\ 5 & 3 \end{pmatrix}$

Show that 5 (A + B) = 5A + 5B

8. Find X and Y if $X + Y = \begin{pmatrix} 7 & 0 \\ 2 & 5 \end{pmatrix}$ and $X - Y = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

[Hints : Adding (X + Y) + (X – Y)]

ANSWERS

2. (a) Not equal, (b) a = 9, b = 6, c = 2, d = –4

3. (i) $\begin{pmatrix} 1 & -1 & 10 \\ 12 & 14 & 5 \\ 2 & -1 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & -5 & 0 \\ 0 & 0 & -11 \\ 2 & -1 & -1 \end{pmatrix}$ (iii) $\begin{pmatrix} 8 & -13 & 5 \\ 6 & 7 & -25 \\ 6 & -3 & -2 \end{pmatrix}$

4. $\begin{pmatrix} 5 & 1 & 7 \\ -1 & 2 & 6 \\ 5 & 16 & 0 \end{pmatrix}$

5. (i) $\begin{pmatrix} 5 & -1 & 1 \\ 3 & -3 & 7 \end{pmatrix}$, (ii) $\begin{pmatrix} -1 & -9 & 1 \\ -7 & 1 & 1 \end{pmatrix}$, (iii) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (iv) $\begin{pmatrix} -6 & -31 & 3 \\ -26 & 5 & 0 \end{pmatrix}$ (v) $\begin{pmatrix} -16 & 40 & -8 \\ 16 & 8 & -32 \end{pmatrix}$

6. (a) 15 (b) $\begin{pmatrix} -5/2 & -1/2 & 0 & -5/2 \\ -1/2 & 5 & -7/2 & -3 \end{pmatrix}$ (c) $\begin{pmatrix} -2 & 2 & -7 \\ -4 & -6 & -3 \end{pmatrix}$

8. $X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

Multiplication of Matrices

Multiplication (or product) of two matrices A and B is possible if and only if the number of columns of A is equal to the number of rows of B.

$$\text{Let } A = \begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 5 & 1 \end{pmatrix}_{3\times 2}, B = \begin{pmatrix} 4 & 0 \\ 3 & 8 \end{pmatrix}_{2\times 2}$$

The product AB is possible because the number of columns of matrix A is equal to the number of rows of the matrix B. The dimension of the product AB will be 3×2.

Definition : If $A = (aij)$ be a $m \times n$ matrix and $B = (bik)$ be an $n \times p$ matrix, then the product AB is defined as the matrix $C = (cik)$ of the type $m \times p$ where

$$Cik = \sum_{j=1}^{n} aij \times bjk$$

Rule of Multiplication of Two Matrices : The rule of the multiplication of the matrices is row–column wise ($\rightarrow \downarrow$). The first row of AB is obtained by multiplying the Ist row with Ist, 2nd and 3rd column of B. Similarly, second row of AB is obtained by multiplying the 2nd row of A with Ist, 2nd and 3rd column of B respectively and so on.

Example 10. If $A = \begin{pmatrix} 2 & 3 & 2 \\ 4 & 3 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 9 \\ 0 & 1 \\ 6 & 9 \end{pmatrix}$, find AB.

Solution : Since A is 2×3 and B is 3×2, then AB is defined.

$$\therefore AB = \begin{pmatrix} 2 & 3 & 2 \\ 4 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 9 \\ 0 & 1 \\ 6 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times 1+3\times 0+2\times 6 & 2\times 9+3\times 1+2\times 9 \\ 4\times 1+3\times 0+1\times 6 & 4\times 9+3\times 1+1\times 9 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 39 \\ 10 & 48 \end{pmatrix}_{2\times 2}$$

Example 11. If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$

Compute AB and BA and show that $AB \neq BA$.

Solution : Since A is 2×3 and B is 3×2, then AB is defined.

$$\therefore AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 2+(-2)\times 4+3\times 2 & 1\times 3+(-2)\times 5+3\times 1 \\ (-4)\times 2+2\times 4+5\times 2 & (-4)\times 3+2\times 5+5\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

Since B is 3×2 and A is 2×3, then BA is also defined.

$$\therefore BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 1+3\times(-4) & 2\times(-2)+3\times 2 & 2\times 3+3\times 5 \\ 4\times 1+5\times(-4) & 2\times(-2)+5\times 2 & 4\times 3+5\times 5 \\ 2\times 1+1\times(-4) & 2\times(-2)+1\times 2 & 2\times 3+1\times 5 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

Hence, $AB \neq BA$

Example 12. If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 & -! \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

Show that AB is a null matrix and BA is not a null matrix.

Solution : $AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

$$= \begin{bmatrix} -1+6-5 & -2+12-10 & -1+6-5 \\ -2-18+20 & -4-36+40 & -2-18+20 \\ -3-12+15 & -6-24+30 & -3-12+15 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ AB = 0 i.e. AB is a null matrix.

$$BA = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1-4-3 & -1+6+2 & 1-8-3 \\ 6+24+18 & 6-36-12 & -6+48+18 \\ 5+20+15 & 5-30-10 & -5+40+15 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 7 & -10 \\ 48 & -42 & 60 \\ 40 & -35 & 50 \end{bmatrix} \neq 0$$

Hence BA is not a null matrix.

Properties of Matrix Multiplication

(i) Matrix multiplication, in general, is not commutative, i.e.

$AB \neq BA$

(ii) Matrix multiplication is associative i.e.

$A(BC) = (AB)C$

where A, B, C are any three matrices of dimension $m \times n$, $n \times p$, $p \times q$ respectively.

(iii) Matrix multiplication is distributive i.e.

$A(B + C) = AB + AC$

where A, B, C are any three matrices of dimension $m \times n$, $n \times p$ and $n \times p$ respectively.

Example 13. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Show that (i) $A(BC) = (AB)C$

(ii) $A(B - C) = AB - AC$

Solution : $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 3+8 & 1+10 \\ 9+16 & 2+20 \end{bmatrix} = \begin{bmatrix} 11 & 11 \\ 25 & 23 \end{bmatrix}$$

$BC = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 3+2 & 3+2 \\ 4+10 & 4+10 \end{bmatrix} = \begin{bmatrix} 5 & 14 \\ 5 & 14 \end{bmatrix}$$

Now, $A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 14 \\ 5 & 14 \end{bmatrix} = \begin{bmatrix} 5+28 & 5+28 \\ 15+56 & 15+26 \end{bmatrix}$

$$= \begin{bmatrix} 33 & 33 \\ 71 & 71 \end{bmatrix} \quad \text{(i)}$$

$(AB)C = \begin{bmatrix} 11 & 11 \\ 25 & 23 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 11+22 & 11+22 \\ 25+46 & 25+46 \end{bmatrix}$

$$= \begin{bmatrix} 33 & 33 \\ 71 & 71 \end{bmatrix} \quad \text{(ii)}$$

From (i) and (ii), $A(BC) = (AB)C$

(ii) $B - C = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix}$

$A(B - C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2+4 & 0+6 \\ 6+8 & 0+12 \end{bmatrix}$

$$= \begin{bmatrix} 6 & 6 \\ 14 & 12 \end{bmatrix} \quad \text{(iii)}$$

$$AC = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\2 & 2\end{bmatrix} = \begin{bmatrix}1+4 & 1+4\\3+8 & 3+8\end{bmatrix} = \begin{bmatrix}5 & 6\\11 & 11\end{bmatrix}$$

$$AB = \begin{bmatrix}11 & 11\\25 & 23\end{bmatrix} \text{ as calculated above.}$$

$$\therefore AB - AC = \begin{bmatrix}11 & 11\\25 & 23\end{bmatrix} - \begin{bmatrix}5 & 6\\11 & 11\end{bmatrix}$$

$$= \begin{bmatrix}6 & 6\\14 & 12\end{bmatrix} \qquad \text{(iv)}$$

Form (iii) & (iv), we have

$$A(B - C) = AB - AC$$

Example 13 A : Show by an example that $AB \neq BA$

or

Show by an example that multiplication of matrices is not commutative.

Solution : Let $A = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$, $B = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

$$AB = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$$

$$BA = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}$$

Thus, $AB \neq BA$

Example 13 B : $AB = 0$ does not necessarily imply that $A=0$ or $B = 0$. Show with an example.

or

If A and B are two matrices such that $AB = 0$, it is not always that $A = 0$ or $B = 0$ or both are equal to zero. Prove it.

Solution : Let $A = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \neq 0 \quad B = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \neq 0$

$$\text{Then } AB = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = 0$$

Thus $AB = 0$ does not necessarily imply that $A = 0$ or $B = 0$

EXERCISE 1.2

1. (a) When is matrix multiplication defined ? Or Define multiplication of matrices.

(b) $A = \begin{bmatrix} 1 & 2 & -3 & 4 & 3 \\ 0 & 4 & 5 & 2 & 4 \\ 5 & 1 & 0 & -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \\ 9 & 10 \end{bmatrix}$

Compute AB . Is BA defined ?

2. If $A = \begin{bmatrix} 2 & -1 & 3 \\ -3 & 2 & 0 \\ 5 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 & -1 \\ 0 & 5 & 2 \\ 1 & -2 & 1 \end{bmatrix}$

Compute AB and BA and show that $AB \neq BA$

3. If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$, $C = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & -2 \\ -3 & -3 & 3 \end{bmatrix}$

Show that AB and CA are null matrices but BA and AC are not null matrices.

4. (a) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Show that $A (B + C) = AB + AC$

(b) If $A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ 3 & 5 \end{pmatrix}$, Find a and b such that

$AB = BA$. Compute $3A + 5B$

Answers

1. (b) $AB = \begin{bmatrix} 47 & 54 \\ 87 & 102 \\ -33 & -32 \end{bmatrix}$, BA or defined 2. $AB = \begin{bmatrix} -3 & -7 & -1 \\ 9 & 4 & 7 \\ -16 & 17 & -4 \end{bmatrix}$, $BA = \begin{bmatrix} -17 & 6 & -8 \\ -5 & 12 & -2 \\ 13 & -4 & 2 \end{bmatrix}$, $AB \neq BA$

4. (b) a = 6.5, b 1.5, $\begin{bmatrix} 41.5 & 13.5 \\ 27 & 28 \end{bmatrix}$

Positive Integral Power of a Square Matrix

Let A is any square matrix, then

$A.A = A^2$

$A^2.A\ (A.A)\ A = A^3 \quad A^2.A = (A.A)\ A = A^3$

$A^3.A = A^4$ etc. $A^3.A = A^4$ etc.

Some Special Types of Matrices

(1) Idempotent Matrix : A square matrix A is said to be idempotent if $A^2 = A$. For example,

$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is an idempotent matrix as in this case $A^2 = A$

(2) Nilpotent Matrix : A square matrix A is said to be nilpotent if $A^n = 0$ for some positive integer n. For example,

$A = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$ is nilpotent of index 2 as $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(3) Involutory Matrix : A square matrix A is said to be involutary if its square is equal to the identity matrix i.e. $A^2 = I$. For example,

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an involutory matrix as in this case $A^2 = I$

Example 14. Show that the following matrix is idempotent :

$$A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

Solution : The matrix A is idempotent if $A^2 = A$

$$A^2 = A.A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3-5 & -3-9+15 & -5-15+25 \\ -1-3+5 & 3+9-15 & 5+15-25 \\ 1+3-5 & -3-9+15 & -5-15+25 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} = A$$

Hence the matrix A is idempotent

Example 15. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2 - 4A - 5I = 0$

where I is unit matrix, 0 is a null matrix.

Solution : $A^2 = A.A = \begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix} = \begin{bmatrix}9 & 8 & 8\\8 & 9 & 8\\8 & 8 & 9\end{bmatrix}$

$$A^2 - 4A - 5I = \begin{bmatrix}9 & 8 & 8\\8 & 9 & 8\\8 & 8 & 9\end{bmatrix} - 4\begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix} - 5\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}9 & 8 & 8\\8 & 9 & 8\\8 & 8 & 9\end{bmatrix} - \begin{bmatrix}4 & 8 & 8\\8 & 4 & 8\\8 & 8 & 4\end{bmatrix} - \begin{bmatrix}5 & 0 & 0\\0 & 5 & 0\\0 & 0 & 5\end{bmatrix}$$

$$= \begin{bmatrix}9-4-5 & 8-8-0 & 8-8-0\\8-8-0 & 9-4-5 & 8-8-0\\8-8-0 & 8-8-0 & 9-4-5\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} = 0$$

$\therefore A^2 - 4A - 5I = 0$

Example 16. If $A = \begin{pmatrix}3 & -4\\1 & -1\end{pmatrix}$, show that $A^n = \begin{pmatrix}1+2n & -4n\\n & 1-2n\end{pmatrix}$

Solution : $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1+2.1 & -4.1 \\ 1 & 1-2.1 \end{pmatrix}$

$$A^2 = A.A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 1+2.2 & -4.2 \\ 2 & 1-2.2 \end{pmatrix}$$

$\therefore$ The result is true for $n = 1, 2$

Let us assume that the result is true for $n = m$ i.e.

$$A^m = \begin{bmatrix} 1+2m & -4m \\ m & 1-2m \end{bmatrix}$$

then $A^{m+1} = A^m . A = \begin{bmatrix} 1+2m & -4m \\ m & 1-2m \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1+2m(m+1) & -4(m+1) \\ m+1 & 1-2(m+1) \end{bmatrix}$$

$\therefore$ The result is true for $n = m + 1$

$\therefore$ The result follows by induction.

EXERCISE 1.3

1. (a) Define (i) Idempotent matrix (ii) Nilpotent matrix (iii) Involutory matrix with suitable examples.

(b) If $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$, Show that $A^2 - 3A + 2I = 0$

where 0 is null matrix and I is unit matrix.

2. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, show that $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

3. Show that the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

satisfies the equation $A^3 - 6A^2 + 9A - 4I = 0$

4. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, Find f (A) where $f(x) = x^2 - 5x + 6$

5. If $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$,

verify that $(A + B)^2 \neq A^2 + 2AB + B^2$

Answers

4. $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & 10 \\ -5 & 4 & 4 \end{bmatrix}$

Transpose of a Matrix

Let A be any matrix. The matrix obtained by interchanging rows and columns of A is called the transpose of A and is denoted by A' or A^T (read as A transpose). Thus if A = [aij] is an m × n matrix, then A' = [aij] will be a n × m matrix.

For example, If $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$, then $A' = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}_{3\times 2}$

Properties of transpose of matrices

(i) The transpose of a transposed matrix is equal to the matrix itself i.e.

$(A')' = A$

(ii) The transpose of the sum of the two matrices is equal to the sum of the transpose the matrices, i.e.

$(A + B)' = A' + B'$

(iii) The transpose of product of two matrices is equal to the product of their transposes in the reverse order i.e.

$(AB)' = B' A'$

Some Special Types of Matrices

(1) Symmetric Matrix : A square matrix A is said to be symmetric matrix if the transpose of A is equal to matrix itself i.e. A' = A. For example, the matrix

$\begin{bmatrix} 1 & 5 & 4 \\ 5 & 2 & 7 \\ 4 & 7 & 3 \end{bmatrix}$ is a symmetric matrix of order 3 as in this case A' = A.

Note : In a symmetric matrix any element $a_{ij} = a_{ji}$ for all values of i and j.

(2) Skew–Symmetric Matrix : A square matrix A is said to be skew–sysmmetric if the transpose of A is equal to – A i.e. A' = – A. For example.

$\begin{bmatrix} 0 & 3 & 5 \\ -3 & 0 & 4 \\ -5 & -4 & 0 \end{bmatrix}$ is a skew–symmetric matrix

Note : In a skew symmetric matrix all the diagonal elements are zero.

(3) Orthogonal matrix : A square matrix A is said to be orthogonal matrix if AA' = I = A'A where A' is transpose of A and I is an identity matrix. For example,

$A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is orthogonal as in this case A A' = I = A'A

Example 17. If $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}$

Show that (AB)' = B' A'

Solution : $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$, $A' = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 0 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}$ $\quad B' = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1+2-2 & 2+0+2 \\ 2+2+0 & 4+0+0 \end{pmatrix} = \begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix} \quad \text{.....(i)}$$

$$B'A' = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1+2-2 & 2+2+0 \\ 2+0+2 & 4+0+0 \end{pmatrix} = \begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix} \quad \text{.....(ii)}$$

From (i) and (ii), (AB)' = B'A'

Example 18. If $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 4 & 3 \\ 4 & 1 \end{pmatrix}$

Show that (ABC)' = C' B' A'

Solution : Given : $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 4 & 3 \\ 4 & 1 \end{pmatrix}$

$$\therefore \quad A' = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, B' = \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}, C' = \begin{pmatrix} 4 & 4 \\ 3 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 2+6 & 5+2 \\ 4+9 & 10+3 \end{pmatrix} = \begin{pmatrix} 8 & 7 \\ 13 & 13 \end{pmatrix}$$

$$\therefore \quad ABC = \begin{pmatrix} 8 & 7 \\ 13 & 13 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 32+28 & 24+7 \\ 52+52 & 39+13 \end{pmatrix} = \begin{pmatrix} 60 & 31 \\ 104 & 52 \end{pmatrix}$$

$$(ABC)' = \begin{pmatrix} 60 & 104 \\ 31 & 52 \end{pmatrix} \quad \text{....(i)}$$

Now $C'B' = \begin{pmatrix} 4 & 4 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 8+20 & 12+4 \\ 6+5 & 9+1 \end{pmatrix} = \begin{pmatrix} 28 & 16 \\ 11 & 10 \end{pmatrix}$$

$$C'B'A' = \begin{pmatrix} 28 & 16 \\ 11 & 10 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 28+32 & 56+48 \\ 11+20 & 22+32 \end{pmatrix} = \begin{pmatrix} 60 & 104 \\ 31 & 52 \end{pmatrix} \quad \text{....(ii)}$$

$\therefore$ From (i) and (ii), $(ABC)' = C'B'A'$

EXERCISE 1.4

1. (a) Define transpose of a matrix. What are its properties ?

   (b) Define (i) Symmetric matrix (ii) Skew–symmetric matrix and (iii) Orthogonal matrix.

   (c) If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 0 \\ 7 & 8 \end{pmatrix}$

   Show that (i) $(A + B)' = A' + B'$

   (ii) $(AB)' = B'A'$

2. Show that $(AB)' = B'A'$ if

$$A = \begin{pmatrix} 0 & 0 & 2 \\ -1 & 2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 0 & -2 \end{pmatrix}$$

3. If $A = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix}$, show that

   (i) $A + A'$ is symmetric (ii) $A - A'$ is skew–symmetric

4. Given that

$$A = \begin{pmatrix} 5 & 6 \\ 6 & 3 \end{pmatrix}, B = \begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix}, C = \begin{pmatrix} 4 & 3 \\ 4 & 1 \end{pmatrix}$$

   Show that $(ABC)' = C'B'A'$

5. Show that the matrix $= \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ is orthogonal

Determinants

Definition : The determinant is a number associated to a square matrix. Determinants are possible only for square matrices. If A is a square matrix, then the determinant of A is denoted by

|A| or det. A or Δ

(i) Determinant of Order One : Let $A = [a_{ij}]$ be a matrix of order 1, then the determinant is defined as :

$$|A| = |a_{ij}| = a_{ij}$$

(ii) Determinant of Order Two :

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a square matrix of order 2, then its det. is defined as :

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}.\, a_{22} - a_{21}.\, a_{12}$$

For example, if $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$, then

$$|A| = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = 18 - 20 = -2$$

Rule : The value of the determinant of order 2 is obtained by multiplying the two elements on the principal diagonal and subtracting the product of the elements on the cross diagonal.

(iii) Determinant or Order Three

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{23} \end{bmatrix}$ be a square matrix of order 3, then its det. is defined as :

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Calculation of determinant :

Expanding it along the first row :

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\,(a_{22}\,a_{33} - a_{32}\,a_{23}) - a_{12}\,(a_{21}\,a_{23} - a_{23}\,a_{31}) + a_{12}\,(a_{21}\,a_{32} - a_{31}\,a_{22})$$

$$= a_{11}.\, a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{12}\,a_{21}\,a_{32} - a_{11}\,a_{23}\,a_{32}.\, a_{12}\,a_{21}\,a_{23} - a_{13}\,a_{22}\,a_{31}$$

Rule : The value of the determinant is obtained by multiplying each element of the first row by a determinant obtained by deleting from the given determinant, the row and the column to which element belongs, the signs being taken positive and negative alternatively.

Note : We have expanded the above determinant by first row. In fact, we can expand the given determinant with respect to the IInd, IIIrd row, Ist column, II column or III column. In case of 3×3 determinant, we can expand in $3 + 3 = 6$ ways. However, case should be taken to attach appropriate sign to various terms of an expansion. As a general rule, it should be remembered that the sign of aij element appearing before any sub–determinant in the expansion above is given by $(-1)^{i+j}$

Example 1. Find the value of the determinant $\begin{vmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Solution : Expanding it along the first row

$$\begin{vmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 4 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1 (45 - 48) - 2 (36 - 42) + 4 (32 - 35)$$

$$= -3 + 12 - 12 = -3$$

Aliter : We get the same result if we expand it along first column :

$$\begin{vmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 8 & 9 \end{vmatrix} + 7 \begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix}$$

$$= 1 (45 - 43) - 4 (18 - 32) + 7 (12 - 20)$$

$$= -3 + 56 - 56 = -3$$

Example 2 : Evaluate the determinant with respect to first column :

$$\begin{vmatrix} 0 & -1 & 2 \\ -2 & 1 & 0 \\ 3 & -3 & -1 \end{vmatrix}$$

Solution : $\begin{vmatrix} 0 & -1 & 2 \\ -2 & 1 & 0 \\ 3 & -3 & -1 \end{vmatrix} = 0 \begin{vmatrix} 1 & 0 \\ -3 & -1 \end{vmatrix} - (-2) \begin{vmatrix} -1 & 2 \\ -3 & -1 \end{vmatrix} + 3 \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix}$

$$= 0 (-1 - 0) + 2 (1 + 6) + 3 (0 - 2)$$

$$= 0 + 14 - 6 = 8$$

Minors and Cofactors

(a) Minors : Let A be a square matrix of order 3, then minor of an element a_{11} is the determinant of the submatrix obtained from A by deleting first row and first column containing the elements a_{11}.

Consider determinant of order 3 :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{23} \end{vmatrix}$$

The minor of the element a_{11} is obtained by deleting first row and first column containing the element a_{11}. It is denoted by M_{11} and is given by :

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Similarly, minor of a_{12} is obtained as :

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

(b) Cofactors : The cofactor Cij of an element a_{ij} is defined as :

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Where M_{ij} is the minor of an element aij

Thus, the minor of an element with a proper sign is called the cofactor of the element.

Consider the determinant of order 3 :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{23} \end{vmatrix}$$

Cofactor of a_{11} is

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Similarly, cofactor of a_{12} is

$$C_{12} = (-1)^{1+2} M_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Expansion of 3rd order Determinant in terms of Cofactors

Using the concept of minor and cofactor, we can state the following rule for the expansion of a 3rd order determinant :

$$\text{Let } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{23} \end{vmatrix}$$

$$= a_{11}.\, C_{11} + a_{12}.\, C_{12} + a_{13}\, C_{13}$$

$$= a_{11}\,(-1)^{1+1}\, M_{11} + a_{12}\,(-1)^{1+2}\, M_{12} + a_{13}\,(-1)^{1+3}\, M_{13}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\,(a_{12}.\, a_{33} - a_{32}.\, a_{22}) - a_{12}\,(a_{21}.\, a_{33} - a_{31}.\, a_{23}) + a_{13}\,(a_{21}.\, a_{32} - a_{31}.\, a_{22})$$

Note : The expansion of the given determinant can be done along any row or column.

Difference between Determinant and Matrix

(i) A matrix may be rectangular or a square but the determinant is always square.

(ii) A matrix has no definite value but a determinant has a definite value.

(iii) We use square brackets [ ] to denote matrix but we use two vertical lines || to denote for determinant.

Example 19. Evaluate the following determinants :

$$\text{(i)} \begin{vmatrix} 3 & 4 & -2 \\ 2 & -3 & 4 \\ -4 & 2 & 3 \end{vmatrix} \quad \text{(ii)} \begin{vmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \\ 2 & 1 & 2 \end{vmatrix} \quad \text{(iii)} \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

Solution : (i) Expanding along the Ist Row, we get

$$\begin{vmatrix} 3 & 4 & -2 \\ 2 & -3 & 4 \\ -4 & 2 & 3 \end{vmatrix} = 3 \begin{vmatrix} -3 & 4 \\ 2 & 3 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ -4 & 3 \end{vmatrix} + (-2) \begin{vmatrix} 2 & -3 \\ -4 & 2 \end{vmatrix}$$

$$= 3(-9-8) - 4(6+16) - 2(4-12)$$
$$= 51 - 88 + 16 = -123$$

(ii) Expanding along the Ist Row, we get

$$\begin{vmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \\ 2 & 1 & 2 \end{vmatrix} = 1 \begin{vmatrix} -1 & 3 \\ 1 & 2 \end{vmatrix} - 3 \begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} + 4 \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= 1(-2-3) - 3(4-6) + 4(2+2)$$
$$= -5 + 6 + 16 = 17$$

(iii) Expanding it along Ist Row, we get

$$\begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix} = 1 \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} - 2 \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} + (-2) \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix}$$

$$= 1(3-0) - 2(-1-0) - 2(2-0)$$
$$= 3 + 2 - 4 = 1$$

Example 20. Find the cofactors of all the elements of the matrix :

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 4 \\ 0 & 5 & 6 \end{bmatrix}$$ Hence evaluate det. A.

Solution : Given $A = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 4 \\ 0 & 5 & 6 \end{bmatrix}$

$$|A| = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 4 \\ 0 & 5 & 6 \end{bmatrix}$$

The cofactors of the elements of the Ist Row in |A| :

$$+\begin{vmatrix}1 & 4\\ 5 & 6\end{vmatrix}, \quad -\begin{vmatrix}2 & 4\\ 0 & 6\end{vmatrix}, \quad \begin{vmatrix}2 & 1\\ 0 & 5\end{vmatrix}$$

i.e. $(6-20), \quad -(12-0), \quad (10-0)$

i.e. $-14, \quad -12, \quad 10$

The cofactors of the elements of the 2nd Row in |A| are :

$$-\begin{vmatrix}2 & 1\\ 5 & 6\end{vmatrix}, +\begin{vmatrix}5 & 1\\ 0 & 6\end{vmatrix}, -\begin{vmatrix}5 & 2\\ 0 & 5\end{vmatrix}$$

i.e. $-(12-5), +(30-0), -(25-0)$

i.e. $-7, \quad 30, \quad -25$

The cofactors of the elements of the 3rd Row in |A| are :

$$\begin{vmatrix}2 & 1\\ 1 & 4\end{vmatrix}, -\begin{vmatrix}5 & 1\\ 2 & 4\end{vmatrix}, +\begin{vmatrix}5 & 2\\ 2 & 1\end{vmatrix}$$

i.e. $(8-1), \quad -(20-2), \quad (5-4)$

i.e. $7, \quad -18, \quad 1$

Thus, cofactor matrix = C (A) = $\begin{bmatrix}-14 & -12 & 10\\ -7 & 30 & -25\\ 7 & -18 & 1\end{bmatrix}$

Evaluation of Determinant |A| in terms of Cofactors

$|A| = 5\,C_{11} + 2\,C_{12} + 1\,C_{13}$

$= 5(-14) + 2(-12) + 1(10)$

$= -70 - 24 + 10 = -84$

EXERCISE 1.5

1. (a) Define determinant of a matrix.
   (b) State the difference between matrix and determinant.
   (c) Define minors and cofactors.
   (d) Evaluate the following determinants :

   (i) $\begin{vmatrix} 2 & 4 \\ 3 & 7 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & 4 \\ 9 & -7 \end{vmatrix}$

2. Evaluate the following determinants :

   (i) $\begin{vmatrix} 0 & 0 & 1 \\ 4 & 2 & 1 \\ 9 & -3 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 4 \\ 6 & 1 & 1 \end{vmatrix}$ (iii) $\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$

3. Find the value of x if

   $$\begin{vmatrix} 1 & 2 & 5 \\ 1 & x & 5 \\ 3 & -1 & 2 \end{vmatrix} = 0$$

4. Find cofactors of all elements of the following matrices

   (i) $\begin{bmatrix} -3 & 4 \\ 7 & 8 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 & 4 \\ 4 & 1 & 2 \\ 3 & 2 & 2 \end{bmatrix}$

5. Evaluate the determinant of the following matrix by developing minors in terms of 2nd column:

$$\begin{bmatrix} 5 & -1 & 4 \\ 0 & -2 & 1 \\ 4 & 1 & 0 \end{bmatrix}$$

ANSWERS

1. (d) (i) 2. (ii) – 57 2. (i) – 30 (ii) 39 (iii) – 2 3. 2 4. (i) $\begin{bmatrix} 8 & -4 \\ -7 & -3 \end{bmatrix}$ (ii) $\begin{vmatrix} -2 & -2 & 5 \\ 2 & -10 & 7 \\ 2 & 14 & -11 \end{vmatrix}$ 5. 23.

Properties of Determinants

The several important properties of the determinants are :

(1) If the rows and columns of a determinant are interchanged, the value of the determinant remains unchanged i.e. |A| = |A'|

Example : Let $A = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$, $|A| = \begin{vmatrix} 7 & 8 \\ 9 & 10 \end{vmatrix} = 70 = 72 = -2$

$$A' = \begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix}, \quad |A'| = \begin{vmatrix} 7 & 8 \\ 9 & 10 \end{vmatrix} = 70 - 72 = -2$$

Thus, $|A| = |A'|$

So the value of $|A|$ and $|A'|$ are the same.

(2) If any two adjacent rows (or columns) of a determinant are interchanged, the value of the determinant changes only in sign.

Example : Let $A = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$, then $|A| = 70 - 72 = -2$

Determinant after interchanging rows

i.e. $|B| = \begin{vmatrix} 9 & 10 \\ 7 & 8 \end{vmatrix}$, $|B| = 72 - 70 = +2$

$\therefore \quad |B| = -|A|$

(3) If any two rows or two columns of a determinant are identical or are multiples of each other, then the value of the determinant is zero.

Example : Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix}$, then

Now $|A| = 1(20 - 12) - 2(16 - 6) + 4(8 - 5)$

$= 1(8) - 2(10) + 4(3)$

$= 8 - 20 + 12$

$= 0$ (Here R_1 and R_3 are identical)

(4) If all the elements of any row or column of a determinant are zero, then the value of the determinant is zero.

Example : Let $|A| = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix}$, then

$$|A| = 0\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} - 0\begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 0$$

(5) If all the elements of any row (or column) of a determinant are multiplied by a quantity (K), the value of the determinant is multiplied by the same quantity.

Example : Let $|A| = \begin{vmatrix} 7 & 4 \\ 2 & 1 \end{vmatrix} = 7 - 8 = -1$

If we multiply the first row by 3 (i.e. $R_1 \times 3$), the new determinant is

$$|B| = \begin{vmatrix} 21 & 12 \\ 2 & 1 \end{vmatrix} = 3\begin{vmatrix} 7 & 4 \\ 2 & 1 \end{vmatrix}$$

Making use of the above property we have

$|B| = k\,|A|$

(6) If each element of a row (or column) of a determinant is sum of two elements, the determinant can be expressed as the sum of two determinants of the same order.

Example : Let $|A| = \begin{vmatrix} a_1 + \alpha_1 & b_1 & c_1 \\ a_2 + \alpha_2 & b_2 & c_2 \\ a_3 + \alpha_3 & b_3 & c_3 \end{vmatrix}$

Then $|A| = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} + \begin{pmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{pmatrix} = |A_1| + |A_2|$

(7) The addition of a constant multiple of one row (or column) to another row (or column) leave the determinant unchanged.

Then $A = \begin{pmatrix} a_1 + \lambda b_1 & b_1 & c_1 \\ a_2 + \lambda b_2 & b_2 & c_2 \\ a_3 + \lambda b_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

(8) The determinant of the product of two matrices of the same order is equal to the product of individual determinants.

i.e. $|ABI = |AI. |BI$

Example : Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$, verify that $|ABI = |AI . |BI$

$$AB = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 2+14 & 6+18 \\ 1+14 & 3+18 \end{bmatrix} = \begin{bmatrix} 16 & 24 \\ 15 & 21 \end{bmatrix}$$

$$\text{Thus, } |ABi = \begin{vmatrix} 16 & 24 \\ 15 & 21 \end{vmatrix} = 16 \times 21 - 15 \times 24 = 336 = -24$$

$$|AI = \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 4 - 2 = 2 \; |BI = \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = 9 - 21 = -12$$

Therefore, $|AI . |BI = 2 (-12) = -24$

$\therefore$ $|ABI = |AI. |BI$

3.10 Symbols and Termionology used in Determinants

R_1, R_2, R_3.....etc. stand for 'first row', 'second row', 'third row'.....etc.

C_1, C_2, C_3.....etc. stand for 'the first column', 'the second column', 'the third column'... ...etc.

Operate $R_2 \to R_2 - R_1$ means 'from the element of second row subtract the corresponding elements of the first row'.

Operate $C_2 \to C_1 + 2C_3$ means 'to the elements of the first column, add twice the corresponding elements of the third column'.

The following examples illustrate the applications of properties of determinants :

Example 1. Evaluate (i) $\begin{vmatrix} 2 & 5 & 8 \\ -3 & 7 & 1 \\ 2 & 5 & 8 \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 6 & -6 & 2 \\ 3 & 1 & -9 & 7 \\ 5 & 0 & -15 & 8 \\ 6 & 3 & -18 & 1 \end{vmatrix}$ using properties of determinants

Solution (i) Since first and third rows are identical, the value of the determinant is zero **(Property III)**

$$\therefore |A| = \begin{vmatrix} 2 & 5 & 8 \\ -3 & 7 & 1 \\ 2 & 5 & 8 \end{vmatrix} = 0$$

(ii) Since the elements of the third columns are equi–multiples (– 3 times) of the elements of the first column, the value of the determinant is zero **(Property IV).**

$$|A| = \begin{vmatrix} 2 & 6 & -6 & 2 \\ 3 & 1 & -9 & 7 \\ 5 & 0 & -15 & 8 \\ 6 & 3 & -18 & 1 \end{vmatrix} = 0$$

Example 2. Prove without expansion that :

$$\begin{vmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 13 & 14 & 15 \end{vmatrix} = 0$$

Solution : Let $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 13 & 14 & 15 \end{vmatrix}$

Operating $C_2 \to C_2 - C_1$; $C_3 \to C_3 - C_2$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 6 & 1 & 1 \\ 13 & 1 & 1 \end{vmatrix}$$

$= 0$ (C_1 and C_3 are identical)

Example 3. Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$

Solution : Operating $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$$

Taking out $(a - b)$ common from C_1 and $(b - c)$ from C_2

$$\Delta = (a - b)(b - c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$$

Expanding it along R_1

$$\Delta = (a - b)(b - c) \left[1 \begin{vmatrix} 1 & 1 \\ a+b & b+c \end{vmatrix} \right]$$

$= (a - b)(b - c)[(b + c) - (a + b)]$
$= (a - b)(b - c)(c - a)$

Example 4. Prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

Solution : Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

Operating $C_1 \to C_1 - C_2$; $C_2 \to C_3 - C_3$ we get

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a-b)(a^2+ab+b^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix}$$

Take out (a – b) and (b – c) common from C_1 and C_2

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ (a^2+ab+b^2) & (b^2+bc+c^2) & c^3 \end{vmatrix}$$

Expanding it along R_1

$= (a - b)(b - c)[b^2 + bc + c^2 - a^2 - ab - b^2]$

$= (a - b)(b - c)[(c2 - a2) + b(c - a)]$

$= (a - b)(b - c)[(c - a)(c + a) + b(c - a)]$

$= (a - b)(b - c)[(c - a)(c + a + b)]$

$= (a - b)(b - c)(c - a)(a + b + c)$

Example 5. Show that $\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix} = 1 + a + b + c$

Solution : Operating $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix} = \begin{vmatrix} 1+a+b+c & b & c \\ 1+a+b+c & 1+b & c \\ 1+c+b+c & b & 1+c \end{vmatrix}$$

Take out $1 + a + b + c$ from C_1

$$= (1 + a + b + c) \begin{vmatrix} 1 & b & c \\ 1 & 1+b & c \\ 1 & b & 1+c \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$= (1 + a + b + c) \begin{vmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding it along C_1

$$= (1 + a + b + c)\left[1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0 - 0\right]$$

$$= (1 + a + b + c)(1 - 0 - 0) = 1 + a + b + c$$

Example 6. Prove that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Solution : Taking out a, b, c common from C_1, C_2 and C_3

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{b} & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b}+1 & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c}+1 \end{vmatrix}$$

Operating $C_1 \to C_1 + C_2 + C_3$ we get

$$= abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b}+1 & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & \frac{1}{c}+1 \end{vmatrix}$$

Take out $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ common from C_1

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} \\ 1 & \frac{1}{b}+1 & \frac{1}{c} \\ 1 & \frac{1}{b} & \frac{1}{c}+1 \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\{[1-0]-0-0\}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Hence the result is proved.

Example 7. Show that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (b-c)(c-a)(a-b)$

Solution : Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & ca-bc \\ 0 & c-a & ab-bc \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & bc \\ 0 & b-a & ca-bc \\ 0 & c-a & ab-bc \end{vmatrix}$$

Expanding along C_1, we get

$$= \begin{vmatrix} b-a & c(a-b) \\ c-a & b(a-c) \end{vmatrix} = (a-b)(c-a) \begin{vmatrix} -1 & c \\ 1 & -b \end{vmatrix}$$

$$= (a-b)(c-a)(b-c) = (a-b)(b-c)(c-a)$$

Example 8. Show that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

Solution : Operating $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix}$$

Take out 2a + 2b + 2c common from C_1, we get

$$= (2a+2b+2c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & a & c+a+b \end{vmatrix}$$

Expanding along C_1

$$= 2(a+b+c).1 \begin{vmatrix} b+c+a & 0 \\ 0 & c+a+b \end{vmatrix}$$

$$= 2(a+b+c)(a+b+c)^2 = 2(a+b+c)^3$$

Example 9. Show that $\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+acd \\ 1 & c & c^2 & c^3+abd \\ 1 & d & d^2 & d^3+abc \end{vmatrix} = 0$

Solution : Let Δ denote the given determinant which may be expressed as the sum of two determinants.

$$\Delta = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & acd \\ 1 & c & c^2 & abd \\ 1 & d & d^2 & abc \end{vmatrix}$$

Denoting the determinant on RHS by Δ_1 and Δ_2

$\Delta = \Delta_1 + \Delta_2$ (i)

Now consider Δ_2

Operating $R_1 \to aR_1$, $R_2 \to bR_2$, $R_3 \to cR_3$ $R_4 \to dR_4$ and dividing the determinant by abcd, we have,

$$\Delta_2 = \frac{1}{abcd} \begin{vmatrix} a & a^2 & a^3 & abcd \\ b & b^2 & b^3 & abcd \\ c & c^2 & c^3 & abcd \\ d & d^2 & d^3 & abcd \end{vmatrix}$$

Take common abcd from C_4

Take common abcd from C_4

$$= \begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{vmatrix}$$

Taking the fourth column to the position of Ist column

$$\Delta = -\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$$

$= -1\,\Delta_1$

From (i) $\Delta = \Delta_1 + \Delta_2 = 0$

EXERCISE 1.6

1. (a) Mention the important properties of determinants.
   (b) Prove without expansion :

   (i) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$ (ii) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

   [Hints : (i) Applying $C_2 \to C_2 + C_3$ (ii) Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_2$]

2. Show that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$ **[Hints :** Applying $C_1 \to C_1 + C_2 + C_3$]

3. Prove that $\begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = 4abc$ **[Hints :** Applying $R_1 - (R_2 + R_3)$, $C_2 - C_3$]

4. Show that $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$

   [Hints : Applying $C_1 \to C_1 + C_2 + C_3$, $R_3 - 2R_1$]

5. Prove that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

   [Hints : Use Property 6]

6. Prove that $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$

   [Hints : Applying $C_1 \to C_1 + C_2 + C_3$]

7 Prove that $\begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix} = (x+y+z)^3$

[**Hints:** Operate $R_1 \to R_1 + R_2 + R_3$ Taking x+y +z common from R_1 and apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$]

Adjoint of a Square Matrix

Definition : If A is any square matrix, then adjoint of A is defined as the transpose of the matrix obtained by the replacing the element of A by their corresponding cofactors. In other words, the adjoint of a matrix A is the transpose of the cofactor matrix. It is denoted by Adj. A or adj. A

Thus if $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$C(A) = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

Then, Adj. A = Transpose of the cofactor matrix $\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$

$$= \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

Steps to Compute Adjoint of a Matrix

(i) Check that the given matrix A is a square.

(ii) Compute Cofactor Matrix C (A). It is obtained by replacing each element in the matrix by its corresponding cofactor.

(iii) Take the transpose of the Cofactor Matrix. The result would give the adjoint of the matrix.

Example 1. Calculate the adjoint of the matrix :

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution : First we shall find the cofactors of the elements of A in |A|

C_{11} = cofactor of $1 = (-1)^{1+1} . 4 = 4$

C_{12} = cofactor of $2 = (-1)^{1+2} . 3 = -3$

C_{21} = cofactor of $3 = (-1)^{2+1} . 2 = -2$

C_{22} = cofactor of $4 = (-1)^{2+2} . 1 = +1$

$\therefore$ C (A) = cofactor matrix = $\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

Adj. A = transpose of the cofactor matrix = transpose of $\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Aliter : The adjoint of a 2 × 2 matrix can be easily found by :
(1) Interchange the entries on the principal diagonal
(2) Negate the entries on the minor diagonal.

Thus, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then

$$\text{Adj. } A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Example 2. Find the adjoint of the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$$

Solution : The cofactors of the elements of the Ist Row in |A| are :

$$+\begin{vmatrix} 4 & 5 \\ -6 & -7 \end{vmatrix}, -\begin{vmatrix} 3 & 5 \\ 0 & -7 \end{vmatrix}, +\begin{vmatrix} 3 & 4 \\ 0 & -6 \end{vmatrix}$$

i.e. $(-28 + 30), -(-21 - 0), +(-18 - 0)$

i.e. $2, \quad 21, \quad -18$

The cofactors of the elements of the 2nd Row in |A| are :

$$+\begin{vmatrix} 0 & -1 \\ -6 & -7 \end{vmatrix}, -\begin{vmatrix} 1 & -1 \\ 0 & -7 \end{vmatrix}, +\begin{vmatrix} 1 & 0 \\ 0 & -6 \end{vmatrix}$$

i.e. $-(0 - 6), +(-7 - 0), -(-6 - 0)$

i.e. $6, -7, 6$

The cofactors of the elements of the 3rd Row in |A| are :

$$+\begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix}, -\begin{vmatrix} 1 & 4 \\ 3 & 5 \end{vmatrix}, +\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix}$$

i.e. $(0 + 4), -(5 + 3), +(4 - 0)$

i.e. $4, -8, 4$

Adj. A = transpose of the cofactor matrix = transpose of $\begin{bmatrix} 2 & 21 & -18 \\ 6 & -7 & 6 \\ 4 & -8 & 4 \end{bmatrix}$

$$\therefore \text{Adj. } A = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

Important Theorem : If A is a square matrix of order n, then
$A\,(\text{Adj. } A) = (\text{Adj. } A)\,A = |A|.\, I$
where I is the identity martix.

Example 3. If $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$, verify that

$$A\,(\text{Adj. } A) = (\text{Adj. } A)\,A = |A|.\, I,$$ where I is identity matrix

Solution : We have

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{vmatrix} = 1\,(6-3) - 1\,(3+6) + 1\,(-1-4)$$

$$= 3 - 9 - 5 = -11$$

Let us find the cofactors of the elements.

The cofactors of the elements of the Ist Row in |A| are :

$$+\begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix}, -\begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix}, +\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

i.e. $+(6-3), -(3+6), +(-1-4)$

i.e. $3, -9, -5$

The cofactors of the elements of the 2nd Row in |A| are :

$$-\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$$

i.e. $-(3+1), +(3-2), -(-1-2)$

i.e. $-4, 1, 3$

The cofactors of the elements of the 3rd Row in |A| are :

$$+\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix}, +\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

i.e. $(-3-2), -(-3-1), +(2-1)$

i.e. $-5, 4, 1$

$$\text{Adj. } A = \text{Transpose of the cofactor matrix} = \text{transpose of } \begin{bmatrix} 3 & -9 & -5 \\ -4 & 1 & 3 \\ -5 & 4 & 1 \end{bmatrix}$$

$$\therefore \text{Adj. } A = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

Now,

$$A\ (\text{Adj. } A) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & 11 \end{bmatrix} = (-11) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= |A|.\ I$ (where I is identity matrix (i)

$$\text{Also} \quad (\text{Adj. } A)\ A = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & 11 \end{bmatrix} = (-11) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= |A|\ .\ I$ (ii)

From (i) and (ii), we have

$A\ (\text{Adj. } A) = (\text{Adj. } A)\ A = |A|\ .\ I$

Hence the result is proved.

Example 4. Show that the transpose of A is one third of its adjoint, where A is a matrix given as :

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Solution : Firstly, we shall find the adjoint of the matrix.

The cofactors of the elements of the Ist Row in |A| are :

$$+\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix}, -\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix}, +\begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix}$$

i.e. $-3, \quad -6, \quad -6$

The cofactors of the elements of the 2nd Row is |A| are :

$$+\begin{vmatrix} -2 & -2 \\ -2 & 1 \end{vmatrix}, -\begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix}, +\begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix}$$

i.e. $6, \quad 3, \quad -6$

The cofactors of the elements of the 3rd Row in |A| are :

$$-\begin{vmatrix} -2 & -2 \\ 1 & -2 \end{vmatrix}, +\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix}, -\begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix}$$

i.e. 6, −6, 3

$\therefore$ Adj. A = transpose of cofactor matrix = transpose of $\begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}$

$$\therefore \text{Adj } A = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$\text{Transpose of } A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$= \frac{1}{3}(\text{Adj. } A)$$

Thus, transpose of A is one third of its adjoint.

EXERCISE 1.7

1. (a) Define adjoint of a matrix

   (b) Find the adjoint of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

2. Find the adjoint of the following matrices,

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

3. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$, verify that

$A\ (Adj.\ A) = (Adj.\ A)\ A = |A|.\ I$

4. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, show that Adj. $A = 3A'$

ANSWERS

1. (b) $\begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & 1 \end{bmatrix}$ 2. (i) $\begin{bmatrix} -3 & 6 & -3 \\ 3 & -3 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Inverse of a Matrix

Let A be any square matrix of order n. The n–square matrix B of the same order is called the inverse of A if $AB = BA = I$.

The inverse of the matrix A is denoted by A^{-1}, so $B = A^{-1}$

Note : For the inverse of the matrix, the matrix must be **non–singular. A matrix A is said to be non–singular if $|A| \neq 0$. Whereas a matrix is said to be singular if its determinant is equal to zero i.e. $|A| = 0$.**

The inverse of A is calculated by the following formula :

$$A^{-1} = \frac{Adj.\ A}{|A|} \text{ provided } |A| \neq 0$$

Steps to find the inverse of a matrix

(i) Find the value of $|A|$. If $|A| \neq 0$, go to step 2. If $|A| = 0$, we cannot find the inverse of the matrix.

(ii) We shall then find the adjoint of the matrix.

(iii) Apply the following formula to calculate the inverse of A

$$A^{-1} = \frac{Adj.\ A}{|A|} \text{ provided } |A| \neq 0$$

Note : The necessary condition for finding the inverse of a matrix A is that (i) the given matrix is

a square matrix and (ii) the determinant of the matrix is not equal to zero i.e. $|A| \neq 0$.

Example 1. Find the inverse of $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Then, $|A| = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = 0 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$

$= 0\,(2-3) - 1(1-9) + 2\,(1-6) \quad = 0 + 8 - 10 = -2 \neq 0$

Now we proceed to find Adj. A

Cofactors of the elements of the Ist row in |A| are :

$$+\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$

i.e. $(2-3), \quad -(1-9), +(1-6)$

i.e. $\quad -1 \qquad 8 \qquad -5$

Cofactors of the elements of the 2nd Row in |A| are :

$$+\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, + \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}, - \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}$$

i.e. $-(1-2), \quad +(0-6), -(0-3)$

i.e. $\quad 1, \qquad -6, \qquad 3$

Cofactors of the elements of the 3rd Row in |A| are :

$$+\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix}, -\begin{vmatrix}0 & 2\\ 1 & 3\end{vmatrix}, +\begin{vmatrix}0 & 1\\ 1 & 2\end{vmatrix}$$

i.e. $+(3-4), \quad -(0-2), \quad +(0-1)$

i.e. $\quad -1, \qquad 2, \qquad -1$

$\therefore$ Adj. A = transpose of the cofactor matrix = transpose of $\begin{bmatrix}-1 & 8 & -5\\ 1 & -6 & 3\\ -1 & 2 & -1\end{bmatrix}$

$$\therefore \text{ Adj. } A = \begin{bmatrix}-1 & 8 & -5\\ 1 & -6 & 3\\ -1 & 2 & -1\end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{Adj.\,A}{|A|}$$

$$= \frac{1}{-2}\begin{bmatrix}-1 & 1 & -1\\ 8 & -6 & 2\\ -5 & 3 & -1\end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix}1/2 & -1/2 & 1/2\\ -4 & 3 & -1\\ -5/2 & -3/2 & 1/2\end{bmatrix}$$

Example 2. Find the inverse of $A = \begin{bmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}$

Solution : $|A| = \begin{vmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ 1 & 3 & 4\end{vmatrix}$

$$= 1\begin{vmatrix}4 & 3\\ 3 & 4\end{vmatrix} - 3\begin{vmatrix}1 & 3\\ 1 & 4\end{vmatrix} + 3\begin{vmatrix}1 & 4\\ 1 & 3\end{vmatrix}$$

$= 1\,(16-9) - 3\,(4-3) + 3\,(3-4)$

$= 7 - 3 - 3 = 1 \neq 0$

Now we proceed to find Adj. A

The cofactors of the elements of the Ist Row in |A| are :

$$+\begin{vmatrix}4 & 3\\ 3 & 4\end{vmatrix} - 1\begin{vmatrix}1 & 3\\ 1 & 4\end{vmatrix} + \begin{vmatrix}1 & 4\\ 1 & 3\end{vmatrix}$$

i.e. $(16-9), -(4-3), \quad -(3-3)$

i.e. $\quad 7, \qquad -1, \qquad -1$

The cofactors of the elements of the 2nd Row in |A| are :

$$+\begin{vmatrix}3 & 3\\ 3 & 4\end{vmatrix} + \begin{vmatrix}1 & 3\\ 1 & 4\end{vmatrix} + \begin{vmatrix}1 & 3\\ 1 & 3\end{vmatrix}$$

i.e. $-(12-9),\quad +(4-3),\quad -(3-3)$

i.e. $\quad -3,\qquad 1,\qquad 0$

The cofactors of the elements of the 3rd Row in |A| are :

$$+\begin{vmatrix}3 & 3\\ 4 & 3\end{vmatrix} + \begin{vmatrix}1 & 3\\ 1 & 3\end{vmatrix} + \begin{vmatrix}1 & 3\\ 1 & 4\end{vmatrix}$$

i.e $\quad (9-12),\quad -(3-3),\quad +(4-3)$

i.e. $\quad -3,\qquad 0,\qquad 1$

Adj. A = transpose of the cofactor matrix = transpose of $\begin{bmatrix}7 & -1 & -1\\ -3 & 1 & 0\\ -3 & 0 & 1\end{bmatrix}$

$$\therefore \text{Adj. } A = \begin{bmatrix}7 & -3 & -3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{Adj.\,A}{|A|} = \frac{1}{1}\cdot\begin{bmatrix}7 & -3 & -3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix} = \begin{bmatrix}7 & -3 & -3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$$

Some Important Properties of Inverse Matrices

(a) If A and B are non–singular matrices of the same order, then

$$(AB)^{-1} = B^{-1}\,.\,A^{-1}$$

(b) The inverse of the transpose of a matrix is equal to the transpose of the inverse of that matrix i.e.

$$(A')^{-1} = (A^{-1})'$$

(c) The inverse of the inverse of a matrix is the matrix itself i.e.

$$(A^{-1})^{-1} = A$$

Example 3. If $A = \begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}$, $B = \begin{bmatrix}4 & 5\\ 3 & 4\end{bmatrix}$, show that

$$(AB)^{-1} = B^{-1}\,A^{-1}$$

Solution : Here $A = \begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix} \Rightarrow |A| = \begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix} = 6-5 = 1 \neq 0$

and Adj. $A = \begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix}$

$$\therefore A^{-1} = \frac{Adj.\,A}{|A|} = \frac{1}{1}\cdot\begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix} = \begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix}$$

Now $B = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} \Rightarrow |B| = \begin{vmatrix} 4 & 5 \\ 3 & 4 \end{vmatrix} = 16 - 15 = 1 \neq 0$

and Adj. $B = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$,

$$\therefore \quad B^{-1} = \frac{Adj.\,B}{|A|} = \frac{1}{1} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$$

Now $B^{-1} A^{-1} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix}$ (i)

Again $AB = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ 29 & 37 \end{bmatrix}$

$$|AB| = \begin{bmatrix} 11 & 14 \\ 29 & 37 \end{bmatrix} = 11 \times 37 - 14 \times 29 = 407 - 406 = 1$$

Adj. $AB = \begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix}$

$$\therefore (AB)^{-1} = \frac{Adj.\,AB}{|AB|} = \frac{1}{1} = \begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix} = \begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix} \quad \text{(ii)}$$

From (i) and (ii), $(AB)^{-1} = B^{-1} A^{-1}$

Example 4. If $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & -1 & 6 \\ -1 & 5 & 1 \end{bmatrix}$

Prove that $(A')^{-1} = (A^{-1})'$

Solution : Given $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & -1 & 6 \\ -1 & 5 & 1 \end{bmatrix}$

$$A' = \begin{bmatrix} 1 & 3 & -1 \\ 3 & -1 & 5 \\ 4 & 6 & 1 \end{bmatrix}$$

$$|A'| = \begin{bmatrix} 1 & 3 & -1 \\ 3 & -1 & 5 \\ 4 & 6 & 1 \end{bmatrix}$$

$$= 1(-1 - 30) - 3(3 - 20) + (-1)(18 + 4)$$
$$= -31 + 51 - 22 = -2$$

We find the Adj. A'

Cofactors of the elements of the Ist Row in |A'| are :

$$+\begin{vmatrix} -1 & 5 \\ 6 & 1 \end{vmatrix}, -\begin{vmatrix} 3 & 6 \\ -1 & 1 \end{vmatrix}, +\begin{vmatrix} 3 & -1 \\ 4 & 6 \end{vmatrix}$$

$+(-1 - 30), -(3 - 20) + (18 + 4)$

i.e. $-31, \quad +17, \quad 22$

Cofactors of the elements of the 2nd Row in |A'| are :

$$-\begin{vmatrix} 3 & -1 \\ 6 & 1 \end{vmatrix}, +\begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$

i.e. $-(3 + 6) + (1 + 4) - (6 - 12)$

i.e. $-9, \quad 5, \quad 6$

Cofactors of the elements of the 3rd Row in |A'| are :

$$\begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}, -\begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix}, +\begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix}$$

i.e. $(15 - 1), \quad (-5 + 3) + (-1 - 9)$

i.e. $14, \quad -8, \quad -10$

Adj. A' = transpose of cofactor Matrix = transpose of $\begin{vmatrix} -31 & 17 & 22 \\ -9 & 5 & 6 \\ 14 & -8 & -10 \end{vmatrix}$

$$= \begin{vmatrix} -31 & -9 & 14 \\ 17 & 5 & -8 \\ 22 & 6 & -10 \end{vmatrix}$$

$$\therefore (A')^{-1} = \frac{\text{Adj. } A'}{|A'|} = \frac{1}{-2} \begin{vmatrix} -31 & -9 & 14 \\ 17 & 5 & -8 \\ 22 & 6 & -10 \end{vmatrix} \qquad (i)$$

Now we find A^{-1}

$$|A| = \begin{bmatrix} 1 & 3 & 4 \\ 3 & -1 & 6 \\ -1 & 5 & 1 \end{bmatrix}$$

$$= (-1 - 30) - 3(3 + 6) + 4(15 - 1)$$
$$= -31 - 27 + 56 = -2$$

We find the Adj. A

Cofactors of the elements of the Ist Row in |A| are :

$$+\begin{vmatrix} -1 & 6 \\ 5 & 1 \end{vmatrix}, -\begin{vmatrix} 3 & 6 \\ -1 & 1 \end{vmatrix}, +\begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

i.e. $(-1 - 30), -(3 + 6), +(15 - 1)$

$-31 \quad , \quad -9 \quad , \quad 14$

Cofactors of the elements of the 2nd Row in |A| are :

i.e. $(3-20),\ +(1\ \ 4),\ -(5+3)$

$17\ ,\quad 5\ ,\quad -8$

Cofactors of the elements of the 3rd Row in |A| are :

i.e. $(18+4),\ -(6-12),\ +(-1-9)$

$22\qquad 6\qquad -10$

$$\text{Adj. A} = \text{transpose of cofactor Matrix} = \text{transpose of } \begin{bmatrix} -31 & -9 & 14 \\ 17 & 5 & -8 \\ 22 & 6 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} -31 & 17 & 22 \\ -9 & 5 & 6 \\ 14 & -8 & -10 \end{bmatrix}$$

$$\therefore (A)^{-1} = \frac{\text{Adj. A}}{|A|} = \frac{1}{-2}\begin{vmatrix} -31 & 17 & 22 \\ -9 & 5 & 6 \\ 14 & -8 & -10 \end{vmatrix}$$

Taking the tanspose, we have

$$(A^{-1})' = \frac{1}{-2}\begin{bmatrix} -31 & -9 & 14 \\ 17 & 5 & -8 \\ 22 & 6 & -10 \end{bmatrix}$$

From (i) and (ii)

$(A')^{-1} = (A^{-1})'$

Example 5. If $A = \begin{bmatrix} 4 & 7 & 6 \\ 7 & -1 & 8 \\ 3 & 4 & 2 \end{bmatrix}$, find A^{-1} and verify that $A\,A^{-1} = I$

Solution : $|A| = \begin{vmatrix} 4 & 7 & 6 \\ 7 & -1 & 8 \\ 3 & 4 & 2 \end{vmatrix} = 4(-2-32) - 7(14-24) + 6(28+3)$

$\therefore |A| = -136 + 70 + 186 = 120 \neq 0$

Now we find the adjoint of the matrix.

Cofactors of the elements of the Ist Row in |A| are :

$$+\begin{vmatrix} -1 & 8 \\ 4 & 2 \end{vmatrix}, -\begin{vmatrix} 7 & 8 \\ 3 & 2 \end{vmatrix}, +\begin{vmatrix} 7 & -1 \\ 3 & 4 \end{vmatrix}$$

i.e. $(-2-32), -(14-24), +(28+3)$

i.e. $-34, \quad 10, \quad 31$

Cofactors of the elements of the 2nd Row in |A| are :

$$+\begin{vmatrix} 7 & 6 \\ 4 & 2 \end{vmatrix}, -\begin{vmatrix} 4 & 6 \\ 3 & 2 \end{vmatrix}, +\begin{vmatrix} 4 & 7 \\ 3 & 4 \end{vmatrix}$$

i.e. $-(14-24), \quad +(8-18), \quad -(16-21)$

i.e. $10, \quad -10, \quad 5$

Cofactors of the elements of the 3rd Row in |A| are :

$$+\begin{vmatrix} 7 & 6 \\ -1 & 8 \end{vmatrix}, -\begin{vmatrix} 4 & 6 \\ 7 & 8 \end{vmatrix}, +\begin{vmatrix} 4 & 7 \\ 7 & -1 \end{vmatrix}$$

i.e. $(56+6), \quad -(32-42), \quad +(-4-49)$

i.e. $62, \quad 10, \quad -53$

Adj. A = transpose of the cofactor matrix = transpose of $\begin{bmatrix} -34 & 10 & 31 \\ 10 & -10 & 5 \\ 62 & 10 & -53 \end{bmatrix}$

$$\therefore \text{Adj. } A = \begin{bmatrix} -34 & 10 & 62 \\ 10 & -10 & 10 \\ 62 & 10 & -53 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{Adj.\ A}{|A|} = \frac{1}{120} \cdot \begin{bmatrix} -34 & 10 & 62 \\ 10 & -10 & 10 \\ 62 & 10 & -53 \end{bmatrix}$$

Verification :

$$\text{Now } AA^{-1} = \frac{1}{120} \cdot \begin{vmatrix} 4 & 7 & 6 \\ 7 & -1 & 8 \\ 3 & 4 & 2 \end{vmatrix} \begin{bmatrix} -34 & 10 & 62 \\ 10 & -10 & 10 \\ 62 & 10 & -53 \end{bmatrix}$$

$$= \frac{1}{120} \cdot \begin{bmatrix} 120 & 0 & 0 \\ 0 & 120 & 0 \\ 0 & 0 & 120 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore AA^{-1} = I$

Hence the result is proved.

Example 6. Show the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 3A + 2I = 0$ and hence find A^{-1}.

Solution : $A^2 = A.A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix}$

$$\therefore A^2 - 3A + 2I = \begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence the equation is satisfied

To find A^{-1}, we pre multiplying the equation by A^{-1}.

$A^{-1} A^2 - 3 A^{-1} A + 2 A^{-1}. I = 0$ or $A - 3I + 2A^{-1} = 0$

or $2A^{-1} = 3I - A$ or $A^{-1} = \frac{1}{2}.[3I - A]$

$$\therefore A^{-1} = \frac{1}{2}\left\{ \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \right\} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & \frac{1}{2} & -2 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Example 7. If $A = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix}$, Find $(I-A)^{-1}$

where I is a identity matrix

Solution : $I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix}$

$$= \begin{bmatrix} 0.9 & -0.2 \\ -0.3 & 0.6 \end{bmatrix}$$

$$|I - A| = \begin{vmatrix} 0.9 & -0.2 \\ -0.3 & 0.6 \end{vmatrix} = 0.54 - .06 = .48$$

$$\text{Adj. } (I-A) = \begin{bmatrix} 0.6 & 0.2 \\ 0.3 & 0.9 \end{bmatrix}$$

$$\therefore (I - A)^{-1} = \frac{Adj.(I - A)}{|I - A|}$$

$$= \frac{\begin{bmatrix} 0.6 & 0.2 \\ 0.3 & 0.9 \end{bmatrix}}{.48} = \frac{1}{.48} \begin{bmatrix} 0.6 & 0.2 \\ 0.3 & 0.9 \end{bmatrix}$$

EXERCISE 1.8

1. Define the following concepts :
   (i) Inverse of a matrix (ii) Singular matrix and (iii) Non–singular matrix.

2. Find A^{-1} if $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

3. Obtain the inverse of (i) $\begin{bmatrix} 4 & 7 & 6 \\ 7 & -1 & 8 \\ 3 & 4 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -4 \\ -2 & 5 & -6 \end{bmatrix}$

4. Consider the matrix $A = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.2 \end{bmatrix}$, find $(I - A)^{-1}$, where I is the identity matrix.

5. If $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, find A^{-1} and show that $AA^{-1} = I$

6. If $A = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix}$, find $(AB)^{-1}$ and verify that
   $(AB)^{-1} = B^{-1} A^{-1}$

7. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$
   Hence or otherwise find A^{-1}.

8. If $A = \begin{bmatrix} 9 & 7 & 6 \\ 7 & -1 & 8 \\ 3 & 4 & 2 \end{bmatrix}$,
   show that $(A')^{-1} = (A^{-1})'$.

9. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, find a matrix B such that $AB = I$
   [Hints : Taking B as A^{-1}]

ANSWERS

2. $\frac{1}{-3}\begin{bmatrix}-3 & 6 & -3\\ 3 & -3 & 0\\ -2 & 0 & 1\end{bmatrix}$ **3.** (i) $\frac{1}{120}\begin{bmatrix}-34 & 10 & 62\\ 10 & -10 & 10\\ 31 & 5 & -53\end{bmatrix}$ (ii) $\frac{-1}{14}\begin{bmatrix}2 & -5 & 3\\ 20 & -8 & 2\\ 16 & -5 & 3\end{bmatrix}$ **4.** $\frac{1}{.66}\begin{pmatrix}.8 & .2\\ .3 & .9\end{pmatrix}$

5. $-\frac{1}{5}\begin{bmatrix}2 & 3 & -13\\ -1 & -4 & 9\\ -1 & 1 & -1\end{bmatrix}$ **7.** $\begin{bmatrix}2/7 & -1/7\\ 1/7 & 3/7\end{bmatrix}$ **9.** $\frac{1}{18}\begin{bmatrix}1 & -5 & 7\\ 7 & 1 & -5\\ -5 & 7 & 1\end{bmatrix}$

Elementary Row (or Column) Operations

The following three operations performed on a matrix are called elementary row operations :

(i) Interchanging of any two rows : This operation is denoted by $R_i \leftrightarrow R_j$ which means that ith and jth rows are interchanged.

(ii) Multiplication of the elements of any row by a non-zero scalar : Such operations are denoted by $R_i \rightarrow kR_i$ which means that all the elements of the ith row are multiplied by k.

(iii) Addition of a non-zero scalar multiple of the elements of any row to the corresponding elements of the other row : Such operations are denoted by as $R_i \rightarrow R_i + kRj$ which means that k times the jth row is added to the ith row.

Note : These operations can also be applied on the columns by similar manner. The corresponding column operations are denoted by

$C_i \leftrightarrow C_j$; $C_i \rightarrow k\,C_i$; $C_i \rightarrow C_i + kC_j$ respectively

To Find Inverse of Matrix by using Gauss Method or Elemenatary Operations

The Gauss Elimination or Gauss Reduction Method or Elementary operations is an alternative method popularly used for finding the inverse of a matrix.

Working Rule

(i) Set up an augumented matrix with the identity matrix on the right, as follows :

[A/I] where I is the identity matrix and A is the given matrix.

(ii) Row operations are applied on [A/I] until the matrix A on the left is reduced to an identity matrix.

(iii) At that point, the matrix on the right will be the inverse i.e. the corrsponding value of I is A^{-1}.

Example 1 : Find the inverse of

$$A = \begin{bmatrix} 2 & 4 \\ 6 & 1 \end{bmatrix}$$ by Gauss Elimination (or Reduction) Method.

Solution : Set up the angumented matrix with identity matrix on the right, as follows :

$$[A/I] = \left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 6 & 1 & 0 & 1 \end{array}\right]$$

Now, we reduce the matrix A to an identity matrix by using elementary row operations.

Iteration 1.

Operating $R_1 \to \frac{1}{2} R_1$, we get

$$\left[\begin{array}{cc|cc} 1 & 2 & 1/2 & 0 \\ 6 & 1 & 0 & 1 \end{array}\right]$$

Operating $R_2 \to R_2 - 6 R_1$, we get

$$\left[\begin{array}{cc|cc} 1 & 2 & 1/2 & 0 \\ 0 & -11 & -3 & 1 \end{array}\right]$$

Iteration 2

Operating $R_2 \to \frac{-1}{11} R_2$, we get

$$\left[\begin{array}{cc|cc} 1 & 2 & 1/2 & 0 \\ 0 & 1 & 3/11 & -1/11 \end{array}\right]$$

Operating $R_1 \to R_1 - 2 R_2$, we get

$$\left[\begin{array}{cc|cc} 1 & 0 & -1/22 & 2/11 \\ 0 & 1 & 3/11 & -1/11 \end{array}\right]$$

Thus $A^{-1} = \begin{bmatrix} -\frac{1}{22} & \frac{2}{11} \\ \frac{3}{11} & \frac{-1}{11} \end{bmatrix}$

Example 2 : Find the inverse of

$$A = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix}$$ by Gauss Reduction Method

Solution : Set up the augnmented matrix with identity matrix on the right, as follows :

$$[A/I] = \left[\begin{array}{ccc|ccc} 4 & 1 & -5 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 & 1 & 0 \\ 3 & -1 & 4 & 0 & 0 & 1 \end{array}\right]$$

Now, we reduce the matrix A on the left to an identity matrix I by applying the row operations

Iteration 1.

Operating $R_1 \to \frac{1}{4} R_1$ i.e. Multiply row 1 by $\frac{1}{4}$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{4} & -\frac{5}{4} & \frac{1}{4} & 0 & 0 \\ -2 & 3 & 1 & 0 & 1 & 0 \\ 3 & -1 & 4 & 0 & 0 & 1 \end{array}\right]$$

Operating $R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{4} & -\frac{5}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{7}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{7}{4} & \frac{31}{4} & -\frac{3}{4} & 0 & 1 \end{array}\right]$$

Iteration 2

Operating $R_2 \to \frac{2}{7} R_2$ i.e. Multiply row 2 by $\frac{2}{7}$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{4} & -\frac{5}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 1 & -\frac{3}{7} & \frac{1}{7} & \frac{2}{7} & 0 \\ 0 & -\frac{7}{4} & \frac{31}{4} & -\frac{3}{4} & 0 & 1 \end{array}\right]$$

Operating $R_1 \to R_1 - \frac{1}{4} R_2$ and $R_3 \to R_3 + \frac{7}{4} R_2$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{8}{7} & \frac{3}{14} & -\frac{1}{14} & 0 \\ 0 & 1 & -\frac{3}{7} & \frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 7 & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right]$$

Iteration 3

Operating $R_3 \to \frac{1}{7} R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{8}{7} & \frac{3}{14} & -\frac{1}{14} & 0 \\ 0 & 1 & -\frac{3}{7} & \frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 1 & -\frac{1}{14} & \frac{1}{14} & \frac{1}{7} \end{array}\right]$$

Operating $R_1 \to R_1 + \frac{8}{7} R_3$ and $R_2 \to R_2 + \frac{3}{7} R_3$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{13}{98} & \frac{1}{98} & \frac{8}{49} \\ 0 & 1 & 0 & \frac{11}{98} & \frac{31}{98} & \frac{3}{49} \\ 0 & 0 & 1 & \frac{-1}{14} & \frac{1}{14} & \frac{1}{7} \end{array}\right]$$

Thus,

$$A^{-1} = \begin{bmatrix} \frac{13}{98} & \frac{1}{98} & \frac{8}{49} \\ \frac{11}{98} & \frac{31}{98} & \frac{3}{49} \\ \frac{-1}{14} & \frac{1}{14} & \frac{1}{7} \end{bmatrix}$$

EXERCISE 1.10

Using Gauss Reduction Method, or Elementary Operations, find the inverse of the following matrices :

1. $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$ 2. $\begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$ 3. $\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ 5. $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ 6. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

ANSWERS

1. $\begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$ 2. $\begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$ 3. $\begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$

4. $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$ 5. $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ 6. $\frac{-1}{3}\begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$

Rank of a Matrix

Definition : The number 'r' is called the rank of the matrix A if

(i) There exists at least one non-zero minor of order r of A

(ii) Every minor of order (r + 1) of A is zero.

The rank of a matrix A is denoted by p(A).

Steps to Find Rank of a Matrix

(i) The given matrix should be a square matrix. If it is not so, the matrix should be made a square matrix by deleting the extra row or the column.

(ii) Find the determinant of the square matrix given or obtained after deleting extra row or column.

(iii) If determinant of the matrix is zero, then take the sub-matrices of the given matrix. If the determinant of any one of the sub-matrices is not zero, then the order of that sub matrix would be the rank of the given matrix.

Example 51. Find the rank of the matrix :

$$\begin{bmatrix} 3 & 2 & 1 \\ 3 & 4 & 3 \\ -1 & 2 & 2 \end{bmatrix}$$

Solution : Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 3 & 4 & 3 \\ -1 & 2 & 2 \end{bmatrix}$

$\therefore |A| = 3(8-6) - 2(6+3) + 1(6+4)$

$= 6 - 18 + 10 = -2 \neq 0$

Thus, minor of order 3 is not zero.

$\therefore$ The rank of the matrix is 3 i.e.

$\rho(A) = 3$

(b) Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 3 & 4 \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 3 & 4 \end{bmatrix}$

$\therefore |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 3 & 4 \end{vmatrix}$

$= 1(12-15) - 2(8-5) + 3(6-3)$

$= -3 - 6 + 9 = 0$

Thus, minor of order 3 is zero

$\therefore$ The rank of the matrix cannot be 3

Now, we shall take any minor of order 2

$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$

Thus, there exist at least one minor of order 2 which is non–zero.

$\therefore \rho(A) = 2$

Example 2. Find the rank of the matrix :

$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Solution : Given matrix is not a square matrix, so we can delete any one column to obtain a square matrix and then find the minor of order 3.

Case I : Deleting IVth column we get,

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 3 & 4 \end{vmatrix} = 1\begin{vmatrix} 3 & 5 \\ 3 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}$$

$= 1\,(12 - 15) - 2\,(8 - 5) + 3\,(6 - 3)$
$= -3 - 6 + 9 = 0$

Case II : Deleting IIIrd column we get,

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 1 \\ 1 & 3 & 5 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 \\ 3 & 5 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}$$

$= 1\,(15 - 3) - 2\,(10 - 1) + 2\,(6 - 3)$
$= 12 - 18 + 6 = 0$

Case III : Deleting IInd column we get,

$$|A| = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 1 & 4 & 5 \end{vmatrix} = 1\begin{vmatrix} 5 & 1 \\ 4 & 5 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} + 2\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}$$

$= 1\,(25 - 4) - 3\,(10 - 1) + 2(8 - 5)$
$= 21 - 27 + 6 = 0$

Case IV : Deleting Ist column we get,

$$|A| = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 5 & 1 \\ 3 & 4 & 5 \end{vmatrix} = 1\begin{vmatrix} 5 & 1 \\ 4 & 5 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 3 & 5 \end{vmatrix} + 2\begin{vmatrix} 3 & 5 \\ 3 & 4 \end{vmatrix}$$

$= (25 - 4) - 3\,(15 - 3) + 2\,(12 - 15)$

$= 46 - 26 + 6 = 0$

As all minors of order 3 are zero, the rank of the matrix cannot be 3. Now we shall take any minor of order 2

$$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$$

As one minor of order 2 is non–zero.

$\therefore$ The rank of the matrix is 2 i.e. $\rho(A) = 2$

Example 53. Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Solution : Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ [$\because R_1, R_2, R_3$ and R_4 are identical]

Here $|A| = 0$, every minor of order 3 and 2 are zero, but individual elements are non–zero i.e. each minor of order 1 is non–zero.

$\therefore \rho(A) = 1$

EXERCISE 1.10

1. (a) Define rank of a matrix

   (b) Find the rank of the following matrices :

   (i) $\begin{bmatrix} 9 & 5 & 6 \\ 7 & -1 & 8 \\ 3 & 4 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 3 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$

2. Find the rank of the following matrices

   (i) $\begin{bmatrix} 2 & 3 & 2 \\ 3 & 5 & 1 \\ 3 & 4 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 5 \\ 3 & 6 & 6 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

3. Find the rank of the following matrices :

   (i) $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 1 & 6 \\ 2 & 4 & 2 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 5 & 6 \\ 4 & 5 & 6 \\ 4 & 5 & 6 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 2 & 1 \\ 3 & -4 & 3 \\ -1 & 2 & 2 \end{bmatrix}$

4. Find the rank of the following matrices :

(i) $\begin{bmatrix} 2 & 7 & 9 & 0 \\ 0 & 5 & 2 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 2 & 3 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 4 \\ 4 & 8 \\ 1 & 2 \end{bmatrix}$

5. Find the rank of A, B, C where C = AB

$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

ANSWERS

1. (b) (i) $\rho(A) = 3$ (ii) $\rho(A) = 2$ (iii) $\rho(A) = 1$

2. (i) $\rho(A) = 2$ (ii) $\rho(A) = 2$ (iii) $\rho(A) = 3$

3. (i) $\rho(A) = 2$ (ii) $\rho(A) = 1$ (iii) $\rho(A) = 3$

4. (i) $\rho(A) = 2$ (ii) $\rho(A) = 3$ (iii) $\rho(A) = 1$

5. $\rho(A) = 1$, $\rho(B) = 2$, $\rho(C) = 1$

Solution of Simultaneous Linear Equations

Matrix algebra and determinants are useful in solving a set of simultaneous linear equations involving two or three variables. Now, we discuss the methods for getting the solution of such equations.

Consider a system of 3 linear equations :

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

In three unknown x_1, x_2 and x_3.

Keeping in mind the rules of multiplication of matrices and their equality, the above system of three linear equations can be written in the matrix form as :

$$AX = B$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

A is called the coefficient matrix.

A set of values x_1, x_2, and x_3 which simultaneously satisfy all the three equations is called a solution of the above system of equations.

Methods of Solving a System of Linear Equations

There are mainly two methods of solving a system of linear equations which are given below:

(1) Cramer's Rule

(2) Matrix Method.

(1) Cramer's Rule: Gabriel Cramer, a Swiss mathematician has developed a method, popularly known as Cramer's Rule, to solve a system of linear equations with the help of determinants.

Consider the system of 3 linear equations:

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

These equations can be written in the matrix form as:

$AX = B$

where, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

If $|A| \neq 0$, then the solution of the equations using Cramer's Rule is given by

$$x_1 = \frac{|A_1|}{|A|}, \; x_2 = \frac{|A_2|}{|A|}, \; x_3 = \frac{|A_3|}{|A|}$$

Where $|A|$ = determinant of coefficient matrix.

$|A_1|$ = determinant of the matrix obtained from A by replacing the coefficient of x_1 (i.e. first column of the coefficient matrix) viz., a_{11}, a_{21}, and a_{31} by the constants b_1, b_2 and b_3 respectively.

$|A_2|$ = determinant of the matrix obtained from A by replacing the coefficient of x_2 (i.e. second column of the coefficient matrix) viz., a_{12}, a_{22}, and a_{32} by the constants b_1, b_2 and b_3 respectively.

$|A_3|$ = determinant of the matrix obtained from A by replacing the coefficient of x_3 (i.e. third column of the coefficient matrix) viz., a_{13}, a_{23} and a_{33} by the constants b_1, b_2 and b_3 respectively.

The following examples illustrate the Cramer's rule.

Example 25. Solve the following equations by using Cramer's rule:

$2x_1 + 3x_2 = 13$

$x_1 + 7x_2 = 23$

Solution: Write down the given equations in matrix form:

$$AX = B$$

where $A = \begin{pmatrix} 2 & 3 \\ 1 & 7 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $B = \begin{pmatrix} 13 \\ 23 \end{pmatrix}$

$$\text{Now, } |A| = \begin{vmatrix} 2 & 3 \\ 1 & 7 \end{vmatrix} = 14-3 = 11$$

$$|A_1| = \begin{vmatrix} 13 & 3 \\ 23 & 7 \end{vmatrix} = 91-69 = 22$$

$$|A_2| = \begin{vmatrix} 2 & 13 \\ 1 & 23 \end{vmatrix} = 46-13 = 33$$

Using Cramer's rule,

$$x_1 = \frac{|A_1|}{|A|} = \frac{22}{11} = 2$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{33}{11} = 3$$

$$\therefore \; x_1 = 2, \text{ and } x_2 = 3$$

Example 26. Solve the following equations by Cramer's rule:

$$3x_1 + x_2 - x_3 = 5$$
$$x_1 + 4x_2 + 2x_3 = 6$$
$$2x_1 + 3x_2 + x_3 = 4$$

Solution: Write down the equations in the matrix form as

$$AX = B$$

where $A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 4 & 2 \\ 2 & 3 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} 5 \\ 6 \\ 4 \end{pmatrix}$

$$\text{Now } |A| = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$$

$$= 3(4-6) - 1(1-4) - 1(3-8)$$

$$= -6 + 3 + 5 = 2 \neq 0$$

$$|A_1| = \begin{vmatrix} 5 & 1 & -1 \\ 6 & 4 & 2 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= 5\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 6 & 2 \\ 4 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 6 & 4 \\ 4 & 3 \end{vmatrix}$$

$$= 5(4-6) - 1(6-8) - 1(18-16)$$

$$= -10 + 2 - 2 = -10$$

$$|A_2| = \begin{vmatrix} 3 & 5 & -1 \\ 1 & 6 & 2 \\ 2 & 4 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} 6 & 2 \\ 4 & 1 \end{vmatrix} - 5\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix}$$

$$= 3(6-8) - 5(1-4) - 1(4-12)$$

$$= -6 + 15 + 8 = 17$$

$$|A_3| = \begin{vmatrix} 3 & 1 & 5 \\ 1 & 4 & 6 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= 3\begin{vmatrix} 4 & 6 \\ 3 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix} + 5\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$$

$$= 3(16-18) - 1(4-12) + 5(3-8)$$

$$= -6 + 8 - 25 = -23$$

Using Cramer's rule,

$$\therefore\ x_1 = \frac{|A_1|}{|A|} = \frac{-10}{2} = -5$$

$$\therefore\ x_2 = \frac{|A_2|}{|A|} = \frac{17}{2} = 8.5$$

$$\therefore\ x_3 = \frac{|A_3|}{|A|} = \frac{-23}{2} = -11.5$$

$\therefore\ x_1 = -5,\ x_2 = 8.5$ and $x_3 = -11.5$

Example 27. Use the method of determinants (Cramer's rule) to solve the set of equations:

$$3x + 2y - z = 4$$
$$-x - y + 3z = 6$$
$$5x - 3y + z = 2$$

Solution: Write down the equations in the matrix form as

$$AX = B$$

where $A = \begin{pmatrix} 3 & 2 & -1 \\ -1 & -1 & 3 \\ 5 & -3 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}$

$$\text{Now } |A| = \begin{pmatrix} 3 & 2 & -1 \\ -1 & -1 & 3 \\ 5 & -3 & 1 \end{pmatrix}$$

$$= 3(-1 + 9) - 2(-1 - 15) - 1(3 + 5)$$

$$= 24 + 32 - 8 = 48$$

$$|A_1| = \begin{vmatrix} 4 & 2 & -1 \\ 6 & -1 & 3 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= 4(-1 + 9) - 2(6 - 6) - 1(-18 + 2)$$

$$= 24 + 32 - 8 = 48$$

$$|A_2| = \begin{vmatrix} 3 & 4 & -1 \\ -1 & 6 & 3 \\ 5 & 2 & 1 \end{vmatrix}$$

$$= 3(6 - 6) - 4(-1 - 15) - 1(-2 - 30)$$

$$= 0 + 64 + 32 = 96$$

$$|A_3| = \begin{vmatrix} 3 & 2 & 4 \\ -1 & -1 & 6 \\ 5 & -3 & 2 \end{vmatrix}$$

$$= 3(-2 + 18) - 2(-2 - 30) + 4(3 + 5)$$

$$= 48 + 64 + 32 = 144$$

Using Cramer's rule,

$$x = \frac{|A_1|}{|A|} = \frac{48}{48} = 1$$

$$y = \frac{|A_2|}{|A|} = \frac{96}{48} = 2$$

$$z = \frac{|A_3|}{|A|} = \frac{144}{48} = 3$$

$\therefore$ $x = 1$, $y = 2$ and $z = 3$

Example 28. Solve the following system of equations by Cramer's rule:

$$2x_1 - x_2 + 3x_3 = 9$$
$$x_2 - x_3 = -1$$
$$x_1 + x_2 - x_3 = 0$$

Solution: The given equations are written as follows:

$$2x_1 - x_2 + 3x_3 = 9$$
$$0x_1 + x_2 - x_3 = -1$$
$$x_1 + x_2 - x_3 = 0$$

Write down the equations in the matrix form as

$$AX = B$$

where $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} 9 \\ -1 \\ 0 \end{pmatrix}$

Now $|A| = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$

$$= 2(-1 + 1) - (-1)(0 + 1) + 3(0 - 1)$$
$$= 0 + 1 - 3 = -2$$

$$|A_1| = \begin{vmatrix} 9 & -1 & 3 \\ -1 & 1 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$
$$= 9(-1 + 1) - (-1)(1 + 0) + 3(-1 - 0)$$
$$= 0 + 1 - 3 = -2$$

$$|A_2| = \begin{vmatrix} 2 & 9 & 3 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{vmatrix}$$
$$= 2(1 - 0) - 9(0 + 1) + 3(0 + 1)$$
$$= 2 - 9 + 3 = -4$$

$$|A_3| = \begin{vmatrix} 2 & -1 & 9 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix}$$
$$= 2(0 + 1) - (-1)(0 + 1) - 9(0 - 1)$$
$$= 2 + 1 - 9 = -6$$

Using Cramer's rule,

$$x_1 = \frac{|A_1|}{|A|} = \frac{-2}{-2} = 1$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{-4}{-2} = 2$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{-6}{-2} = 3$$

$\therefore\ x_1 = 1,\ x_2 = 2$ and $x_3 = 3$

Example 29. Solve by Cramer's rule

$$x - y - z = 1 = y - z - x = z - x - y$$

Solution: The given equations are written as:

$$x - y - z = 1$$
$$-x + y - z = 1$$
$$-x - y + z = 1$$

Write down the equations in the matrix form as:

$$AX = B$$

where $A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\text{Now } |A| = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$

$$= 1(1 - 1) + 1(-1 - 1) - 1(1 + 1)$$
$$= 0 - 2 - 2 = -4$$

$$|A_1| = \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= 1(1 - 1) + 1(1 + 1) - 1(-1 - 1)$$
$$= 0 + 2 + 2 = 4$$

$$|A_2| = \begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix}$$

$$= 1(1 + 1) - 1(-1 - 1) - 1(-1 + 1)$$
$$= 2 + 2 - 0 = 4$$

$$|A_3| = \begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix}$$

$$= 1(1+1) + 1(-1+1) + 1(1+1)$$

$$= 4$$

Using Cramer's rule,

$$x = \frac{|A_1|}{|A|} = \frac{4}{-4} = -1$$

$$y = \frac{|A_2|}{|A|} = \frac{4}{-4} = -1$$

$$z = \frac{|A_3|}{|A|} = \frac{4}{-4} = -1$$

$\therefore$ $x = -1$, $y = -1$ and $z = -1$

EXERCISE 2.10

1. Solve the following equations by using Cramer's rule:

   $9x_1 + x_2 = 13$

   $8x_1 + 2x_2 = 16$

2. Solve the following equations by Cramer's rule:

   (i) $x + y + z = 1$

   $x + 2y + 3z = 6$

   $x + 3y + 4z = 6$

   (ii) $2x_1 + 3x_2 + x_3 = 0$

   $4x_1 - 8x_2 - 6x_3 = 2$

   $6x_1 + x_2 - x_3 = 0$

3. Solve the following equations with the help of determinants:

   $x + y + z = 6$

   $2x - y + 2z = 6$

   $-x - y = -5$

4. Solve by Cramer's rule the following equations:

   $x - y - z = 1 = y - z - x = z - x - y$

5. If AX = B, where

   $$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

   Find x_1 and x_2, by using Cramer's rule.

Answers

1. $x_1 = 1,\ x_2 = 4$ 2. (i) $x=1, y=-5, z=5$ (ii) $x_1 = -\frac{1}{2}, x_2 = 1, x_3 = -2$

3. $x=3, y=2, z=1$ 4. $x=-1, y=-1, z=-1$ 5. $x_1 = 5,\ x_2 = -2$

(2) Matrix Method: Under this method, we use inverse of the matrix to solve a system of equations.

Consider the system of 3 linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad \text{...(i)}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad \text{...(ii)}$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \quad \text{...(iii)}$$

These equations can be written in the matrix form as:

$$AX = B$$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

If $|A| \neq 0$, then the solution of the equation using matrix inverse method is given by

$$X = A^{-1}B$$

where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, A^{-1} = Inverse of the coeff. matrix, $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

The following examples illustrate the matrix inverse method:

Example 30. Solve the following set of equations by using matrix inverse method:

$$2x_1 + 3x_2 = 5$$

$$11x_1 - 5x_2 = 6$$

Solution: Write down the given equations in the matrix form, we have

$$AX = B$$

where $A = \begin{pmatrix} 2 & 3 \\ 11 & -5 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $B = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

Now $|A| = \begin{vmatrix} 2 & 3 \\ 11 & -5 \end{vmatrix} = -10 - 33 = -43 \neq 0$

Since $|A| \neq 0$, therefore A^{-1} exists. Let us now find A^{-1}.

The cofactors of the elements of the first row in $|A|$ are:

$-5, -11$

The cofactors of the elements of the second row in |A| are :

$$-3, -2$$

$$\text{Thus, } C(A) = \begin{pmatrix} -5 & -11 \\ -3 & 2 \end{pmatrix}$$

$$\text{Adj. } A = [C(A)]' = \begin{pmatrix} -5 & -11 \\ -3 & 2 \end{pmatrix}' = \begin{pmatrix} -5 & -3 \\ -11 & 2 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{\text{Adj. } A}{|A|} = -\frac{1}{43}\begin{pmatrix} -5 & -3 \\ -11 & 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{43} & \frac{3}{43} \\ \frac{11}{43} & \frac{-2}{43} \end{pmatrix}$$

Using Matrix Invese Method

$$\text{Now, } X = A^{-1}B = \begin{pmatrix} \frac{5}{43} & \frac{3}{43} \\ \frac{11}{43} & \frac{-2}{43} \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{5}{43} & \frac{3}{43} \\ \frac{11}{43} & \frac{-2}{43} \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{5}{43} \times 5 + \frac{3}{43} \times 6 \\ \frac{11}{43} \times 5 + \frac{-2}{43} \times 6 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{43}{43} \\ \frac{43}{43} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \quad x_1 = 1, \; x_2 = 1$$

Example 31. Solve the following equations by matrix inverse method :

$$2x + y + z = 1$$

$$x - 2y - z = 1.5$$

$$3y - 5z = 9$$

Solution : The given equation are written as :

$$2x + y + z = 1$$

$$x - 2y - z = 1.5$$

$$0x + 3y - 5z = 9$$

Write down the equations in the matrix form as :

$$AX = B$$

Where $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1.5 \\ 9 \end{pmatrix}$

Now $|A| = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{pmatrix}$

$= 2\ (10 + 3) - 1\ (-5 - 3) + 0\ (-1 + 2)$

$= 26 + 8 + 0 = 34 \neq 0$

Since $|A| \neq 0$, therefore A^{-1} exists. Let us now find A^{-1}.

The cofactors of the elements of the Ist Row of $|A|$ are:

$$+\begin{vmatrix} -2 & -1 \\ 3 & -5 \end{vmatrix}, -\begin{vmatrix} 1 & -1 \\ 0 & -5 \end{vmatrix}, +\begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix}$$

i.e. $(10 + 3), -(-5 - 0), +(3 - 0)$

or 13, 5, 3

The cofactors of the elements of the 2nd Row of $|A|$ are:

$$-\begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix}, +\begin{vmatrix} 2 & 1 \\ 0 & -5 \end{vmatrix}, -\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix}$$

i.e. $-(-5 - 3), +(-10 - 0), -(6 - 0)$

or 8, −10, −6

The cofactors of the elements of the 3rd Row of $|A|$ are:

$$+\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, -\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}, +\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

i.e. $(-1 + 2), -(-2 - 1), +(-4 - 1)$

or 1, 3, −5

$$C(A) = \text{Cofactor Matrix} = \begin{pmatrix} 13 & 5 & 3 \\ 8 & -10 & -6 \\ 1 & 3 & -5 \end{pmatrix}$$

$$\text{Adj. } A = [C(A)]' = \begin{pmatrix} 13 & 5 & 3 \\ 8 & -10 & -6 \\ 1 & 3 & -5 \end{pmatrix}$$

$$\therefore \quad \text{Adj. } A = \begin{pmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{pmatrix}$$

$$\therefore \quad A^{-1} = \frac{Adj.\,A}{|A|} = \frac{1}{34}\begin{pmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{pmatrix}$$

Using matrix inverse method

$$X = A^{-1}B = \frac{1}{34}\begin{pmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 15 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{34}\begin{pmatrix} 34 \\ 17 \\ -51 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \\ -1.5 \end{pmatrix}$$

$\therefore \quad x = 1, y = 0.5, z = -1.5$

Example 32. Solve the following equations by using matrix method :

$$2x - y + 3z = 9$$

$$x + y + z = 6$$

$$x - y + z = 2$$

Solution : Write down the equations in the matrix form as :

$$Ax = B$$

Where $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 9 \\ 6 \\ 2 \end{pmatrix}$

Now $|A| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$

$= 2(1 + 1) - (-1)(1 - 1) + 3(-1 - 1)$

$= 4 - 0 - 6 = -2 \neq 0$

Since $|A| \neq 0$, therefore A^{-1} exists. Let us now find A^{-1}.

The cofactors of the elements of the Ist Row of |A| are :

$\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

i.e. 2, 0, −2

The cofactors of the elements of the 2nd Row of |A| are :

$\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

i.e. 2, 0, −2

The cofactors of the elements of the 2nd Row of |A| are :

$-\begin{vmatrix} -1 & 3 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix}$

i.e. − 2, − 1, 1

The cofactors of the elements of the 3rd Row of |A| are :

$$\begin{vmatrix} -1 & 3 \\ -1 & 1 \end{vmatrix}, \; - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, \; - \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix}$$

i.e. – 4, 1, 3

$$\therefore \; C(A) = \begin{pmatrix} 2 & 0 & -2 \\ -2 & -1 & 1 \\ -4 & 1 & 3 \end{pmatrix}$$

$$\therefore \; \text{Adj. } A = [C(A)]' = \begin{pmatrix} 2 & 0 & -2 \\ -2 & -1 & 1 \\ -4 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{pmatrix}$$

$$\therefore \; A^{-1} = \frac{\text{Adj. } A}{|A|} = \frac{1}{-2} \begin{pmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{pmatrix}$$

Using matrix inverse method

$$X = A^{-1} B = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{pmatrix} \begin{pmatrix} 9 \\ 6 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -9+6+4 \\ 0+3-1 \\ 9-3-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$\therefore \; x = 1, \; y = 2, \; z = 3$

Example 33. Find the inverse of $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and hence solve the equations

$$2x_1 - x_2 = 3$$
$$-x_1 + 2x_2 = -3$$

Solution : $|A| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 \neq 0$

Now we find adj. A

Cofactor of the elements of Ist Row in |A| are

2 1

Cofactor of the elements of 2nd Row in |A| are

1 2

$$C(A) = \text{Cofactor Matrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\text{Adj. } A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{Adj.\,A}{|A|} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Using Matrix Inverse Method,

$X = A^{-1} B$

$$\text{Where } X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, B = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$$

Substituting the values, we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = = \frac{1}{3}\begin{pmatrix} 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\therefore x_1 = 1, x_2 = -1$

EXERCISE 2.11

1. Solve the following equations by matrix inverse method

   $x + y + z = 1$

   $x + 2y + 3z = 6$

   $x + 3y + 4z = 6$

2. Solve the following system of equations using matrix inverse method

   $x + y - z = 3$

   $2x + 3y + z = 10$

   $3x - y - 7z = 1$

3. Solve by matrix inverse the following system of equations

   $2x_1 + 3x_2 + x_3 = 0$

   $4x_1 - 8x_2 - 6x_3 = 2$

   $6x_1 + x_2 - x_3 = 0$

   Verify the result through Cramer's rule :

4. If $A = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $B = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $AX = B$.

Find x_1 and x_2.

5. Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -2 & 3 \\ 1 & 2 & -3 \end{pmatrix}$

Hence or otherwise solve the equations :

$x + 2y + 3z = 11$

$x - 2y + 3z = 3$

$x + 2y - 3z = -1$

Answers

1. $x = 1, y = -5, z = 5$ **2.** $x = 3, y = 1, z = 1$ **3.** $x_1 = -\frac{1}{2}, x_2 = 1, x_3 = -2$

4. $X_1 = \frac{-17}{14}$ $X_2 = \frac{33}{14}$ **5.** $A^{-1} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & -1/4 & 0 \\ 1/6 & 0 & -1/6 \end{pmatrix}$, $x = 1, y = 2, z = 2$

Economic Applications of Matrices

A. Applications of Multiplication of Matrices

Example 34. A manufacturer produces three products A, B and C which he sells in two markets. Annual sales volumes are indicated as follows :

Market	Products		
	A	B	C
I	4,000	18,000	20,000
II	12,000	5,000	3,000

(i) Find the total revenue in each market if sales price of A, B and C are Rs. 2, 1.50 and 1.25 respectively.

(ii) Also find the gross profit given the unit cost for A, B and C to be Rs. 1.80, Rs. 1.00 and 0.80 respectively.

Solution : The sales matrix is of order 2×3. The required revenue matrix is of order 2×1. Therefore we will put the price in the column matrix of order 3×1.

TR = Quantity × Price = Q × P

$$TR = \begin{bmatrix} 4,000 & 18,000 & 20,000 \\ 12,000 & 5,000 & 3,000 \end{bmatrix} \begin{bmatrix} 2 \\ 1.50 \\ 1.25 \end{bmatrix}$$

$$TR = \begin{bmatrix} 4,000 \times 2 + 18,000 \times 1.5 + 20,000 \times 1.25 \\ 12,000 \times + 5,000 \times 1.5 + 3,000 \times 1.25 \end{bmatrix} \begin{bmatrix} 8,000 + 27,000 + 25,000 \\ 24,000 + 7,5000 + 3.750 \end{bmatrix}$$

$$= \begin{bmatrix} 60,000 \\ 35,250 \end{bmatrix}$$

∴ TR in each market is Rs. 60,000 and 35,250 respectively.

TC = Quantity × Average cost = Q × AC

$$TC = \begin{bmatrix} 4,000 & 18,000 & 20,000 \\ 12,000 & 5,000 & 3,000 \end{bmatrix} \begin{bmatrix} 1.80 \\ 1.00 \\ 0.80 \end{bmatrix}$$

$$= \begin{bmatrix} 4,000 \times 1.80 + 18,000 \times 1.00 + 20,000 \times 0.80 \\ 12,000 \times 1.80 + 5,000 \times 1.00 + 3,000 \times 0.80 \end{bmatrix} = \begin{bmatrix} 7,200 + 18,000 + 16,000 \\ 21,000 + 5,000 + 2,400 \end{bmatrix}$$

$$= \begin{bmatrix} 41,200 \\ 29,000 \end{bmatrix}$$

Gross Profit = TR − TC

$$= \begin{bmatrix} 60,000 \\ 35,250 \end{bmatrix} - \begin{bmatrix} 41,200 \\ 29,000 \end{bmatrix} = \begin{bmatrix} 18,800 \\ 6,250 \end{bmatrix}$$

∴ Profits from market I and II are Rs. 18,800 and 6,250.

Example 35. The weekly production figures of a motor corporation having two types of factories are as follows:

	Factory A	*Factory B*
Buses	20	30
Trucks	40	10

The corporation has 5, A type factories and 7, B type factories. Buses and trucks sell at Rs. 50,000 and Rs. 40,000 respectively. Estimate the total weekly production of buses and trucks and the total market value of vehicles produced each week.

Solution: Let *Factory AFactory B*

$$X = \begin{matrix} \text{Buses} \\ \text{Trucks} \end{matrix} \begin{pmatrix} 20 & 30 \\ 40 & 10 \end{pmatrix}_{2 \times 2}$$

$$Y = \begin{matrix} \text{Factory A} \\ \text{Factory B} \end{matrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix}, \quad Z = \begin{matrix} \text{Buses} \\ \text{Trucks} \end{matrix} \begin{pmatrix} 50,000 \\ 40,000 \end{pmatrix}$$

(i) Total weekly production = $XY = \begin{pmatrix} 20 & 30 \\ 40 & 10 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 20 \times 5 + 30 \times 7 \\ 40 \times 5 + 10 \times 7 \end{pmatrix}$

$$= \begin{pmatrix} 100 + 210 \\ 200 + 70 \end{pmatrix} = \begin{pmatrix} 310 \\ 270 \end{pmatrix}$$

∴ Weekly production of buses and trucks are 310 and 270.

(ii) Market value of vehicles $= XZ = \begin{pmatrix} 20 & 30 \\ 40 & 10 \end{pmatrix} \begin{pmatrix} 50{,}000 \\ 40{,}000 \end{pmatrix}$

$= \begin{pmatrix} 10{,}00{,}000+12{,}00{,}000 \\ 2{,}00{,}000+4{,}00{,}000 \end{pmatrix} = \begin{pmatrix} 22{,}00{,}000 \\ 24{,}00{,}000 \end{pmatrix}$

$\therefore$ Total market value of buses = Rs. 22,00,000

$\therefore$ Total market value of trucks = Rs. 24,00,000

Example 36. The Deptt. of Education runs 120 colleges and 15 universities having a strength of 300 lecturers, 100 readers and 50 professors in the universities and 5,000 lecturers and 1,000 readers in colleges. The monthly salary is Rs. 9,000 for professors, 8,000 for readers and 5,000 for lecturers. Find monthly salary bill in the colleges and universities.

Solution: Posts in colleges and universities are as follows:

	Colleges		Universities		Colleges + Universities
Professors	0		50		750
Readers	1000	+ 15	100	=	1,21,500
Lecturers	5000		300		604500

$\Big]_{3\times 1}$

Total salary bill = Total Posts × Salary

$= [750 \quad 1{,}21{,}500 \quad 604500]_{1\times 3} \begin{bmatrix} 9{,}000 \\ 8{,}000 \\ 5{,}000 \end{bmatrix}_{3\times 1}$

$[6750000 + 9{,}72{,}000000 + 3022500000]$

Total Salary Bill = Rs. 4001250000

Example 37. There are two families A and B. In family A, there are 2 men, 3 women and one child. In family B, there are 1 man, 2 women and 2 children. The recommended daily allowance for calories is: Man - 2400, woman - 1900, child - 1800 and Protiens: Man - 55 gm, woman - 45 gm, child - 33 gm. Represent the above information by matrices. Using matrix algebra, calculate the total requirement of calories and protien for each of two familities.

Solution: Let $X = \begin{matrix} A \\ B \end{matrix} \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ (M W C), $Y = \begin{matrix} M \\ W \\ C \end{matrix} \begin{pmatrix} 2400 & 55 \\ 1900 & 45 \\ 1800 & 33 \end{pmatrix}_{3\times 2}$ (Calories, Protein)

$\therefore XY = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 2400 & 55 \\ 1900 & 45 \\ 1800 & 33 \end{pmatrix}$

Calories Protein

$= \begin{matrix} A \\ B \end{matrix} \begin{pmatrix} 4800+5700+1800 & 110+135+33 \\ 2400+3800+3600 & 55+90+66 \end{pmatrix} = \begin{pmatrix} 12300 & 278 \\ 9800 & 211 \end{pmatrix}$

Family A: Calories: 12300 Protein: 278 gm

Family B: Calroles: 9800 Protein: 211 gm

B. Cramer's Rule

Example 38. An automobile company uses three types of steel S_1, S_2, S_3 for producing three types of Cars C_1, C_2, C_3. Steel requirements (in tons) for each type of car and total available steel of all the three types is summarised in the following table:

Steel	C_1	C_2	C_3	Total Steel Available
S_1	2	3	4	29
S_2	1	1	2	13
S_3	3	2	1	16

Determine the number of cars of each type which can be produced using Cramer's rule.

Solution: Let x, y and z denote the number of cars produced of each type. Then we have:

$$2x + 3y + 4z = 29$$
$$x + y + 2z = 13$$
$$3x + 2y + z = 16$$

Putting the above equations in the matrix form as

$$AX = B$$

$$\text{Where} A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 29 \\ 13 \\ 16 \end{pmatrix}$$

$$\text{Now} |A| = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 4 \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 2(1 - 4) - 3(1 - 6) + 4(2 - 3) = 2(-3) - 3(-5) + 4(-1)$$
$$= -6 + 15 - 4 = 5$$

$$|A_1| = \begin{vmatrix} 29 & 2 & 4 \\ 13 & 1 & 2 \\ 16 & 2 & 1 \end{vmatrix} = 29 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 13 & 2 \\ 16 & 1 \end{vmatrix} + 4 \begin{vmatrix} 13 & 1 \\ 16 & 2 \end{vmatrix}$$

$$= 29(1 - 4) - 2(13 - 32) + 4(26 - 16) = 29(-3) - 2(-19) + 4(10)$$
$$= -87 + 57 + 40 = 10$$

$$|A_2| = \begin{vmatrix} 2 & 29 & 4 \\ 1 & 13 & 2 \\ 3 & 16 & 1 \end{vmatrix} = 2 \begin{vmatrix} 13 & 2 \\ 16 & 1 \end{vmatrix} - 29 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 4 \begin{vmatrix} 1 & 13 \\ 3 & 16 \end{vmatrix}$$

$$= 2(13 - 32) - 29(1 - 6) + 4(16 - 39) = 2(-19) - 29(-5) + 4(-23)$$
$$= -38 + 145 - 92 = 15$$

$$|A_3| = \begin{vmatrix} 2 & 2 & 29 \\ 1 & 1 & 13 \\ 3 & 2 & 16 \end{vmatrix} = 2 \begin{vmatrix} 1 & 13 \\ 2 & 16 \end{vmatrix} - 2 \begin{vmatrix} 1 & 13 \\ 3 & 16 \end{vmatrix} + 29 \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 2(16 - 26) - 2(16 - 39) + 29(2 - 3) = 2(-10) - 2(-23) + 29(-1)$$
$$= -20 + 69 - 29 = 20$$

Using Cramer's rule,

$$x = \frac{|A_1|}{|A|} = \frac{10}{5} = 2$$

$$y = \frac{|A_2|}{|A|} = \frac{15}{5} = 3$$

$$z = \frac{|A_3|}{|A|} = \frac{20}{5} = 4$$

Hence the solution is

$x = 2$, $y = 3$, and $z = 4$

Example 39. A salesman has the following record of sales during three months for three items A, B and C which have different rates of commission:

Months	Sales of units			Total commission drawn (in Rs.)
	A	B	C	
January	90	100	20	800
February	130	50	40	900
March	60	100	30	850

Find out the rates of commission (in Rs.) on items A, B and C using Cramer's rule.

Solution: Let x, y and z be the rates of commission (in Rs.) of items A, B and C respectively. Then we have

$$90x + 100y + 20z = 800$$
$$130x + 50y + 40z = 900$$
$$60x + 100y + 30z = 850$$

Putting the above equations in the matrix form as

$$AX = B$$

where $A = \begin{pmatrix} 90 & 100 & 20 \\ 130 & 50 & 40 \\ 60 & 100 & 30 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 800 \\ 900 \\ 850 \end{pmatrix}$

$$\text{Now, } |A| = \begin{vmatrix} 90 & 100 & 20 \\ 130 & 50 & 40 \\ 60 & 100 & 30 \end{vmatrix} = 10 \times 50 \times 10 \begin{vmatrix} 9 & 2 & 2 \\ 13 & 1 & 4 \\ 6 & 2 & 3 \end{vmatrix} \text{ (Taking out common)}$$

$$= 5000 \begin{vmatrix} 9 & 2 & 2 \\ 13 & 1 & 4 \\ 6 & 2 & 3 \end{vmatrix} = 5000 \left[9 \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} - 2 \begin{vmatrix} 13 & 4 \\ 6 & 3 \end{vmatrix} + 2 \begin{vmatrix} 13 & 1 \\ 6 & 2 \end{vmatrix} \right]$$

$$= 50000 \left[9 (3 - 8) - 2 (39 - 24) + 2 (26 - 6) \right]$$

$$= 5000 [-45 - 30 + 40] = 5000 \times (-35) = -175000$$

$$|A_1| = \begin{vmatrix} 800 & 100 & 20 \\ 900 & 50 & 40 \\ 850 & 100 & 30 \end{vmatrix} = 50 \times 50 \times 10 \begin{vmatrix} 16 & 2 & 2 \\ 18 & 1 & 4 \\ 17 & 2 & 3 \end{vmatrix}$$ (Taking out common)

$$= 25000 \left[16 \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} - 2 \begin{vmatrix} 18 & 4 \\ 17 & 3 \end{vmatrix} + 2 \begin{vmatrix} 18 & 1 \\ 17 & 2 \end{vmatrix} \right]$$

$$= 25000 \, [16 (3 - 8) - 2 (54 - 68) + 2 (36 - 17)]$$

$$= 25000 \, [-80 + 28 + 38] = 25000 \times (-14) = -350000$$

$$|A_2| = \begin{vmatrix} 90 & 800 & 20 \\ 130 & 900 & 40 \\ 60 & 850 & 30 \end{vmatrix} = 10 \times 50 \times 10 \begin{vmatrix} 9 & 16 & 2 \\ 13 & 18 & 4 \\ 6 & 17 & 3 \end{vmatrix}$$ (Taking out common)

$$= 5000 \left[9 \begin{vmatrix} 18 & 4 \\ 17 & 3 \end{vmatrix} - 16 \begin{vmatrix} 13 & 4 \\ 6 & 3 \end{vmatrix} + 2 \begin{vmatrix} 13 & 18 \\ 6 & 17 \end{vmatrix} \right]$$

$$= 5000 \, [9 (54 - 68) - 16 (39 - 24) + 2 (221 - 108)]$$

$$= 5000 \, [-126 - 240 + 226] = 5000 \times (-140) = -700000$$

$$|A_3| = \begin{vmatrix} 90 & 100 & 800 \\ 130 & 50 & 900 \\ 60 & 100 & 850 \end{vmatrix} = 10 \times 50 \times 50 \begin{vmatrix} 9 & 2 & 16 \\ 13 & 1 & 18 \\ 6 & 2 & 17 \end{vmatrix}$$ (Taking out common)

$$= 25000 \left[9 \begin{vmatrix} 1 & 18 \\ 2 & 17 \end{vmatrix} - 2 \begin{vmatrix} 13 & 18 \\ 6 & 17 \end{vmatrix} + 16 \begin{vmatrix} 13 & 1 \\ 6 & 2 \end{vmatrix} \right]$$

$$= 25000 \, [9 (17 - 36) - 2 (221 - 108) + 16 (26 - 6)]$$

$$= 25000 \, [-171 - 226 + 320] = 25000 \times (-77) = -1925000$$

Using Cramer's Rule,

$$x = \frac{|A_1|}{|A|} = \frac{-350000}{-175000} = 2$$

$$y = \frac{|A_2|}{|A|} = \frac{-700000}{-175000} = 4$$

$$z = \frac{|A_3|}{|A|} = \frac{-1925000}{-175000} = 11$$

Hence the solution is

$x = 2$, $y = 4$, and $z = 11$

Example 40. The equilibrium condition for two goods is given by

$$5p_1 - 2p_2 = 15$$

$$-p_1 + 8p_2 = 16$$

Find the equilibrium price using Cramer's rule.

Solution: Given:

$$5p_1 - 2p_2 = 15$$
$$-p_1 + 8p_2 = 16$$

Putting the equations in matrix form as:

$$AX = B$$

where $A = \begin{pmatrix} 5 & -2 \\ -1 & 8 \end{pmatrix}$, $X = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$, $B = \begin{pmatrix} 15 \\ 16 \end{pmatrix}$

Now $|A| = \begin{vmatrix} 5 & -2 \\ -1 & 8 \end{vmatrix} = 40 - 2 = 38$

$|A_1| = \begin{vmatrix} 15 & -2 \\ 16 & 8 \end{vmatrix} = 120 + 32 = 152$

$|A_2| = \begin{vmatrix} 5 & 15 \\ -1 & 16 \end{vmatrix} = 80 + 15 = 95$

Using Cramer's rule

$$\therefore \quad p_1 = \frac{|A_1|}{|A|} = \frac{152}{38} = 4 \quad \text{and} \quad p_2 = \frac{|A_2|}{|A|} = \frac{95}{38} = 2.50$$

Example 41. The equilibrium condition for the related markets is given by

$$11p_1 - p_2 - p_3 = 31$$
$$-p_1 + 6p_2 - 2p_3 = 26$$
$$-p_1 - 2p_2 + 7p_3 = 24$$

Find the equilibrium price for each market using Cramer's rule.

Solution: Given:

$$11p_1 - p_2 - p_2 = 31$$
$$-p_1 + 6p_2 - 2p_3 = 26$$
$$-p_1 - 2p_2 + 7p_3 = 24$$

Putting the given equations in matrix form as

$$AX = B$$

$$A = \begin{pmatrix} 11 & -1 & -1 \\ -1 & 6 & -2 \\ -1 & -2 & 7 \end{pmatrix}, \quad X = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \quad B = \begin{pmatrix} 31 \\ 26 \\ 24 \end{pmatrix}$$

Now

$$|A| = \begin{vmatrix} 11 & -1 & -1 \\ -1 & 6 & -2 \\ -1 & -2 & 7 \end{vmatrix} = 11 \begin{vmatrix} 6 & -2 \\ -2 & 7 \end{vmatrix} - (-1) \begin{vmatrix} -1 & -2 \\ -1 & 7 \end{vmatrix} + (-1) \begin{vmatrix} -1 & 6 \\ -1 & -2 \end{vmatrix}$$

$$= 11 (42 - 4) + 1 (-7 - 2) - 1 (2 + 6) = 11 (38) - 9 - 8$$

$$= 418 - 9 - 8 = 401$$

$$|A_1| = \begin{vmatrix} 31 & -1 & -1 \\ 26 & 6 & -2 \\ 24 & -2 & 1 \end{vmatrix} = 31 \begin{vmatrix} 6 & -2 \\ -2 & 7 \end{vmatrix} - (-1) \begin{vmatrix} 26 & -2 \\ 24 & 7 \end{vmatrix} + (-1) \begin{vmatrix} 26 & 6 \\ 24 & -2 \end{vmatrix}$$

$$= 31 (42 - 4) + 1 (182 + 48) - 1 (-52 - 144)$$

$$= 1178 + 230 + 196 = 1604$$

$$|A_2| = \begin{vmatrix} 11 & 31 & -1 \\ -1 & 26 & -2 \\ -1 & 24 & 7 \end{vmatrix} = 11 \begin{vmatrix} 26 & -2 \\ 24 & 7 \end{vmatrix} - 31 \begin{vmatrix} -1 & -2 \\ -1 & 7 \end{vmatrix} + (-1) \begin{vmatrix} -1 & 26 \\ -1 & 24 \end{vmatrix}$$

$$= 11 (182 + 48) - 31 (-7 - 2) - 1 (-24 + 26)$$

$$= 11 \times 230 - 31 (-9) - 1 (+2)$$

$$= 2530 + 279 - 2 = 2807$$

$$|A_3| = \begin{vmatrix} 11 & -1 & 31 \\ -1 & 6 & 26 \\ -1 & -2 & 24 \end{vmatrix} = 11 \begin{vmatrix} 6 & 26 \\ -2 & 24 \end{vmatrix} - (-1) \begin{vmatrix} -1 & 26 \\ -1 & 24 \end{vmatrix} + 31 \begin{vmatrix} -1 & 6 \\ -1 & -2 \end{vmatrix}$$

$$= 11 (144 + 52) + 1 (-24 + 26) + 31 (2 + 6)$$

$$= 2156 + 2 + 248 = 2406$$

Using Cramer's rule,

$$p_1 = \frac{|A_1|}{|A|} = \frac{1604}{401} = 4$$

$$p_2 = \frac{|A_2|}{|A|} = \frac{2807}{401} = 7$$

$$p_3 = \frac{|A_3|}{|A|} = \frac{2406}{401} = 6$$

Thus, $p_1 = 4$, $p_2 = 7$, and $p_3 = 6$

Example 42. Consider the following national income determination model:

$$Y = C + I + G,\ C = a + b(Y - T),\ T = d + ty$$

Solve the model for Y, C and T using Cramer's rule.

Solution: We can rewrite the given system of equations as:

$$Y - C + 0T = I + G$$
$$-bY + C + bT = a$$
$$-tY + 0C + T = d$$

The equations in the matrix form is

$$AX = B$$

where $A = \begin{pmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{pmatrix}$, $X = \begin{pmatrix} Y \\ C \\ T \end{pmatrix}$, $B = \begin{pmatrix} I+G \\ a \\ d \end{pmatrix}$

Now $|A| = \begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} - (-1) \begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} + 0 \begin{vmatrix} -b & 1 \\ -t & 0 \end{vmatrix}$

$= 1(1 - 0) + 1(-b + bt) + 0(0 + t)$

$= 1 - b + bt$

$|A_1| = \begin{vmatrix} I+G & -1 & 0 \\ a & 1 & b \\ d & 0 & 1 \end{vmatrix} = (I+G) \begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} - (-1) \begin{vmatrix} a & b \\ b & 1 \end{vmatrix} + 0 \begin{vmatrix} a & 1 \\ d & 0 \end{vmatrix}$

$= (I+G)(1-0) + 1(a - bd) + 0(0 - d)$

$= I + G + a - bd$

$|A_2| = \begin{vmatrix} 1 & I+G & 0 \\ -b & a & b \\ -t & d & 1 \end{vmatrix} = 1 \begin{vmatrix} a & b \\ d & 1 \end{vmatrix} - (I+G) \begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} + 0 \begin{vmatrix} -b & a \\ -t & d \end{vmatrix}$

$= 1(a - bd) - (I+G)(-b + bt) + 0(-bd + at)$

$= (a - bd) - (I+G)(bt - b)$

$|A_3| = \begin{vmatrix} 1 & -1 & I+G \\ -b & 1 & a \\ -t & 0 & d \end{vmatrix} = 1 \begin{vmatrix} t & a \\ 0 & d \end{vmatrix} - (-1) \begin{vmatrix} -b & a \\ -t & d \end{vmatrix} + (I+G) \begin{vmatrix} -b & 1 \\ -t & 0 \end{vmatrix}$

$= 1(d - 0) - 1(-bd + at) + (I+G)(0 + t)$

$= d + (at - bd) + (I+G)t$

Using Cramer's Rule

$$Y = \frac{|A_1|}{|A|} = \frac{I + G + a - bd}{1 - b + bt}$$

$$Y = \frac{|A_2|}{|A|} = \frac{(a - bd) - (I + G)(bt - b)}{1 - b + bt}$$

$$Y = \frac{|A_3|}{|A|} = \frac{d + (at - bd) + (I + G)t}{1 - b + bt}$$

C. Matrix Inverse Method

Example 43. The daily cost of operating a hospital is a linear function of the number of indoor patients (I) and outdoor patients (O) plus fixed costs (z) i.e. $C = xI + yO + z$. Find the values of costs X, Y and Z through a system of linear equations using matrix method:

Days	C, Costs (in Rs.)	Patients I	Patients O
1	5200	30	20
2	5360	35	22
3	5650	15	45

Solution: Let x denotes the total cost of indoor patients, y the total cost of outdoor patient, Z the fixed cost and C, the total cost. Then we have

$30x + 20y + z = 5200$

$35x + 22y + z = 5360$

$15x + 45y + z = 5650$

Putting the equations in the matrix form as

$AX = B$

$$\text{where} A = \begin{pmatrix} 30 & 20 & 1 \\ 35 & 22 & 1 \\ 15 & 45 & 1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 5200 \\ 5360 \\ 5650 \end{pmatrix}$$

$$\text{Now} |A| = \begin{vmatrix} 30 & 20 & 1 \\ 35 & 22 & 1 \\ 15 & 45 & 1 \end{vmatrix} = 30 \begin{vmatrix} 22 & 1 \\ 45 & 1 \end{vmatrix} - 20 \begin{vmatrix} 35 & 1 \\ 15 & 1 \end{vmatrix} + 1 \begin{vmatrix} 35 & 22 \\ 15 & 45 \end{vmatrix}$$

$= 30\ (22 - 45) - 20\ (35 - 15) + 1\ (1575 - 330)$

$= -690 - 400 + 1245 = 155$

Since $|A| \neq 0$, therefore A^{-1} exists. Let us now find A^{-1}

The cofactors of the elements of the Ist Row of $|A|$ are:

$$+ \begin{vmatrix} 22 & 1 \\ 45 & 1 \end{vmatrix}, - \begin{vmatrix} 35 & 1 \\ 15 & 1 \end{vmatrix}, + \begin{vmatrix} 35 & 22 \\ 15 & 45 \end{vmatrix}$$

i.e. $(22 - 45),\ -(35 - 15),\ +(1575 - 330)$

i.e. $-23,\ -20,\ +1245$

The cofactors of the elements of the 2nd Row of $|A|$ are:

$$- \begin{vmatrix} 20 & 1 \\ 45 & 1 \end{vmatrix}, - \begin{vmatrix} 30 & 1 \\ 15 & 1 \end{vmatrix}, - \begin{vmatrix} 30 & 20 \\ 15 & 45 \end{vmatrix}$$

i.e. $-(20 - 45),\ +(30 - 15),\ -(1350 - 300)$

i.e. $+25,\ +15,\ -1050$

The cofactors of the elements of the 3rd Row of |A| are:

$$+\begin{vmatrix} 20 & 1 \\ 22 & 1 \end{vmatrix}, -\begin{vmatrix} 30 & 1 \\ 35 & 1 \end{vmatrix}, +\begin{vmatrix} 30 & 20 \\ 35 & 22 \end{vmatrix}$$

i.e. $+(20 - 22), \quad -(30 - 35), \quad +(660 - 700)$

i.e. $-2, \quad +5, \quad -40$

$$\therefore \quad C(A) = \begin{bmatrix} -23 & -20 & +1245 \\ +25 & +15 & -1050 \\ -2 & +5 & -40 \end{bmatrix}$$

$$\therefore \quad Adj.\,A = [C(A)]' = \begin{pmatrix} -23 & -20 & +1245 \\ +25 & +15 & -1050 \\ -2 & +5 & -40 \end{pmatrix}' = \begin{pmatrix} -23 & 25 & -2 \\ -20 & 15 & 5 \\ 1245 & -1050 & -40 \end{pmatrix}$$

$$A^{-1} = \frac{Adj.\,A}{|A|} = \frac{\begin{bmatrix} -23 & 25 & -2 \\ -20 & 15 & 5 \\ 1245 & -1050 & -40 \end{bmatrix}}{155}$$

$$A^{-1}B = \frac{1}{155}\begin{bmatrix} -23 & 25 & -2 \\ -20 & 15 & 5 \\ 1245 & -1050 & -40 \end{bmatrix}\begin{bmatrix} 5200 \\ 5360 \\ 5650 \end{bmatrix}$$

$$= \frac{1}{155}\begin{bmatrix} -23 \times 5200 + 25 \times 5360 - 2 \times 5650 \\ -20 \times 5200 + 15 \times 5360 + 5 \times 5650 \\ 1245 \times 5200 - 1050 \times 5360 - 40 \times 5650 \end{bmatrix}$$

$$= \frac{1}{155}\begin{bmatrix} -119600+134000-11300 \\ -104000+80400+28250 \\ 6474000-5628000-226000 \end{bmatrix} = \frac{1}{155}\begin{bmatrix} 3100 \\ 4650 \\ 620000 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \\ 4000 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \\ 4000 \end{pmatrix} \therefore \quad x = 20,\ y = 30,\ z = 4000$$

Example 44. The input-output coefficient matrix for 2-sector economy is

$$A = \begin{bmatrix} 0.40 & 0.25 \\ 0.20 & 0.50 \end{bmatrix}$$

The final demand for two industries are 18 and 44 units respectively, Find the gross output (Q) if $Q = (I-A)^{-1}F$ [I = Identity matrix; F = Final Demand vector)

Solution: $I-A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.40 & 0.25 \\ 0.20 & 0.50 \end{bmatrix} = \begin{bmatrix} 0.60 & -0.25 \\ -0.20 & 0.50 \end{bmatrix}$

Now $|I - A| = .60 \times .50 - (-.25 \times -.20) = 0.30 - .05 = 0.25$

$$\text{Adj. } (I-A) = \begin{bmatrix} 0.50 & 0.25 \\ 0.20 & 0.60 \end{bmatrix}$$

$$\therefore \ (I-A)^{-1} = \frac{\text{Adj. } (I-A)}{|I-A|} = \frac{1}{0.25} \cdot \begin{bmatrix} 0.50 & 0.25 \\ 0.20 & 0.60 \end{bmatrix}$$

$$X = (I-A)^{-1} \cdot F = \frac{1}{0.25} \cdot \begin{bmatrix} 0.50 & 0.25 \\ 0.20 & 0.60 \end{bmatrix} \begin{bmatrix} 18 \\ 44 \end{bmatrix}$$

$$= \frac{1}{0.25} \begin{bmatrix} 9+11 \\ 3.6+26.4 \end{bmatrix} = 4 \begin{bmatrix} 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 80 \\ 120 \end{bmatrix}$$

Hence the gross output of two sectors are 80 and 120.

EXERCISE 2.12

1. A manufacturer produces three products A, B and C which he sells in two markets. Annual sale volumes are indicated as follows:

Market	Products		
	A	B	C
I	10,000	2,000	18,000
II	6,000	20,000	8,000

Find the total revenue in each market if the sale price of A, B and C are Rs. 2.50, 1.25 and 1.5 respectively.

2. In a certain city there are 50 colleges and 400 schools. Each school and college has 18 peons, 5 clerks and 1 cashier. Each college in addition has one section officer and one librarian. The monthly salary of each of them is as follows:

Peon - Rs. 300, Clerk - Rs. 500, Cashier - Rs. 600, Section officer - Rs. 700 and Librarian - Rs. 900.

Calculate the total monthly salary bill of colleges and schools together by using matrix algebra.

3. A firm produces three products A, B and C which it sell in two markets. Annual sales in units are given below:

Market	Units sold		
I	8000	4000	16000
II	7000	18000	9,000

If the price per unit of A, B and C are Rs. 2.50, Rs. 1.25 and Rs. 1.50 and the costs per unit are Rs. 1.70, Rs. 1.20 and Rs. 0.80, find the total profit in each market by using matrix algebra.

4. Given the following National Income Model:

$C = a + by$ $\quad (a > 0,\ 0 < b < 1)$
$I = d + ey$ $\quad (d > 0,\ 0 < e < 1)$
$Y = C + I$

Solve for endogenous variables C, I and Y using matrix method or cramer's rule.

5. Given the price data for Apples, Potatoes and Onions in Rs. per kg. and the expenditure for October, November and December to be Rs. 21, 18 and 20.5 respectively. If the family buys the same combination of these in each, find the quantities of these commodities that the family buys using Cramer's rule:

	Apples	Potatoes	Onions
October	5	1	2
November	3	1.5	1.5
December	4	1	2.5

6. The equilibrium condition for three related markets are given by the equations:

$3p_1 - p_2 + p_3 = 2$
$-15p_1 + 6p_2 - 5p_3 = 5$
$5p_1 - 2p_2 + 2p_3 = 1$

Find the equilibrium price for each market using matrix method or Cramer's rule.

7. Three products X, Y, Z are produced after being processed through three departments D_1, D_2 and D_3. The following data are available:

	Products (hours required for a unit produced)		
	In D_1	In D_2	In D_3
X	2	5	1
Y	1	2	3
Z	2	2	3
Max. time available in hours	1100	1800	1400

Find by Cramer's rule the number of units produced for each product to have full utilisation of capacity.

8. A, B and C has Rs. 480, Rs. 760 and Rs. 710 respectively. They utilised the amounts to purchase three types of shares of prices x, y and z respectively. A purchased 2 shares of price x, 5 of price y and 3 of price z. B purchases 4 shares of price x, 3 of price y and 6 of price z. C purchases 1 share of price x, 4 of price y and 10 of price z. Find x, y and z.

9. The total cost C for a manufacturer producing two items X_1 and X_2 is given by

$$C = a_0 + a_1 x_1 + a_2 x_2$$

Data for three months is as follows :

Month	Cost (in Rs.)	No. of Items Produced X_1	X_2
1	6,950	40	10
2	6,725	35	9
3	7,100	40	12

Determine a_0, a_1 and a_2 using Cramer's rule or matrix inverse method

10. An amount of Rs. 10,000 is put into three investments at the rate of interest of 10, 12 and 15 percent per annum. The combined income is Rs. 1,310 and the combined income from the first and second investment is Rs. 190 short of the income from the third. Find the investment in each, using Cramer's rule.

[**Hint :** $x + y + z = 10{,}000$

$$\frac{10}{100}x + \frac{12y}{100} + \frac{15}{100}z = 1310$$

$$\frac{10}{100}x + \frac{12}{100}y - \frac{15}{100}z = -190$$

11. A company produces three products every day. Their total production on a certain day is 45 tons. It is found that the production of the third product exceeds the production of the first product by 8 tons, while the total production of the first and the third product is twice the production of the second product. Determine the production level of each product using Cramer's rule or Matrix inverse method.

[**Hint :** $x + y + z = 45$

$z = x + 8$

$x + z = 2y$

12. To control a certain crop disease, it is necessary to use 8 units of chemical A, 14 units of chemical B and 13 units of chemical C. One barrel of spray P contains one unit of A, 2 units of B and 3 units of C. One barrel of spray Q contains 2 units of A, 3 units of B and 2 units of C. One barrel of spray R contains one unit of A, 2 units of B and 2 units of C. Find, using matrix inverse method, how may barrels of each type of spray be used to just meet the requirement.

[**Hint :** $x + 2y + z = 8$

$2x + 3y + 2z = 14$

$3x + 2y + 2z = 13$

13. A amount of Rs. 5,000 is put into three different investments at the rates of interest of 6%, 7% and 8% per annum respectively. The total annual income is Rs. 358. If the income from the first two investments is Rs. 70 more than the income from the third, find the amount of each investment by using matrix algebra.

[Hint : $x + y + z = 5000$

$$\frac{6}{100}x + \frac{7}{100}y + \frac{8}{100}z = 358$$

$$\frac{6}{100}x + \frac{7}{100}y - \frac{8}{100}z = 70,$$

14. The technological matrix for an I/O model is :

$$A = \begin{pmatrix} 0.1 & 0.2 \\ 0.3 & 0.2 \end{pmatrix},$$

Find $(I - A)^{-1}$ i.e. Leontief inverse.

Answers

1. Rs. 54500, Rs. 52000 2. Rs. 3905000 3. Rs. 17,800, Rs. 12,800
4. $C = \frac{a - ae + bd}{1 - e - b}$, $I\frac{d + ea - bd}{1 - e - b}$ and $Y = \frac{d + a}{1 - e - b}$
5. Apples = 2, Potatoes = 5, Onions = 3 6. $p_1 = 3$, $p_2 = 15$, $p_3 = 8$
7. $x = 200$, $y = 100$, $z = 300$ 8. $x = 11090/119$, $y = 3400/119$ $z = 5980/119$
9. $a_0 = 5000$, $a_1 = 30$, $a_2 = 75$ 10. Rs. 2000, Rs. 3000, Rs. 5000 11. 11, 15, 19
12. $x = 1$, $y = 2$, $z = 3$ 13. Rs. 1000, Rs. 2200, Rs. 1800 14. $\frac{1}{.66}\begin{pmatrix} .8 & .2 \\ .3 & .9 \end{pmatrix}$

UNIVERSITY QUESTIONS

1. Explain the following terms :
   (i) Non–Singular matrix (ii) Symmetrix matrix (iii) Scalar matrix (iv) Inverse of a matrix
   (v) Transpose of a matrix (vi) Determinant.

2. (a) Define and explain by examples :
   (i) Rank of a matrix (ii) Identity matrix (iii) Trace of a matrix (iv) Orthogonal matrix
   (v) Idempotent matrix (vi) Adjoint of a matrix

3. The equilibrium conditions for three related markets are given by the following equations :
   $3p_1 = 2p_2 - 3p_3 + 2$
   $6p_2 = 17p$ $_3 - 2$
   p

n prices and quantities of two commodities market models with

$= -2 - q$; $x_{d_2} = -3 - p - q$ and $x_{s_2} = -9 + p + q$

he quantity.

es for each market by martrix inverse method.

n using (i) Cramer's rule, and (ii) Matrix method :

6. (a) Show that $\begin{vmatrix} 0 & 1 & 1 \\ 1 & a+b & b \\ 1 & a & b \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & a & 0 \\ 1 & a & b \end{vmatrix}$

   Hint : Operating $C_2 \to C_2 - bC_1$; $C_3 \to C_3 - bC_1$; $R_3 \to R_3 + bR_1$

   (b) If $A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 0 \\ 2 & -8 \\ 8 & 16 \end{bmatrix}$

   obtain $A^{-1}B$, and $B'(A^{-1})'$ where B' is transpose of B.

7. Solve the equations by using Cramer's rule or matrix inverse method :
   $x + y + z = 6$, $2x + 5y + 5z = 27$, $2x + 5y + 11z = 45$

8. If $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, prove that $A^2 = 4A + 5I$

9. Matrix A has x – rows and x + 3 columns; and matrix B has y – rows and 9 – y columns. Both AB and BA defined. Find x and y.

10. A tyre manufacturing company uses three types of rubber R_1, R_2, R_2 for manufacturing types of typres T_1, T_2, T_3. The rubber requirement in tons for each type of tyre is given by following matrix :

	T_1	T_2	T_3
R_1	3	4	5
R_2	2	2	3
R_3	4	3	2

Determine the number of tyres which can be produced from 310, 180 and 215 tons of rubber of each type.

Answers

3. $p_1 = 3$; $p_2 = 14$; $p_3 = 7$ 4. $p = 2$, $q = 1$

5. $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ 6. (a) $\begin{bmatrix} 4.5 & 2 \\ 1.75 & 7 \\ 3.75 & -1 \end{bmatrix}$; (b) $\begin{bmatrix} 4.5 & 1.75 & 3.75 \\ 2 & 7 & -1 \end{bmatrix}$

7. $x = 1$, $y = 2$, $z = 3$ 9. $x = 3$, $y = 6$ 10. $x = 20$, $y = 25$, $z = 30$.